Lecture Notes on Cooperative Game Theory

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based on lectures given by Prof. Dr. Stef Tijs †on his visit to METU in November 2006

1 Introduction to Cooperative Game Theory

Outline

1. Introduction
2. Cooperative games. Examples
3. The Shapley value
4. Imputations. The core
5. Convex games
6. Total big boss games
7. ORG's (Operations Research Games (lin.prod., sequencing, flow, mstg))
8. Other solutions (τ-value, nucleolus,...)
9. Other models (Bargaining games, NTU-games,...)

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1.1 Introduction

Game Theory:

- Non-cooperative game theory
  Question: How to behave optimally in an interactive situation?

  1. Games in extensive form ((SPNE) sub game perfect Nash equilibria)
  2. Games in normal form ((NE) Nash equilibria and refinements and equilibrium selections)

  1928 minimax theorem (J. von Neumann), 1944 GEB (J. von Neumann and Morgenstern), 1950 NE (J. Nash), $1965, 1975$ SPNE (R.Selten), 1968 BE (J. Harsanyi).

- Cooperative game Theory (binding agreements, side payments)
  Question: With whom to cooperate?
  Question: How to share earnings, costs?
  1928 J. von Neumann, 1944 stable sets (J. von Neumann and O. Morgernstern), 1953 Shapley value, 1954 Core (Gillies), 1969 Nucleolus(Schmeidler), 1980 τ-value (Tijs), Kernel, Bargaining set.

1.2 Cooperative Games-Examples

$N = 1, 2, ..., n$: set of *players*
2^N : subsets of N *coalitions*
$v : 2^N \to \mathbb{R}$ with $v(\phi) = 0$: *characteristic function* of game $< N, v >$
$v(S)$: *worth* of S (amount which can be obtained when players in S work together).
Game $< N, v >$ or game v.

EXAMPLE 1.1 *(glove game)* $N = 1, 2, 3$. *Players* 1 *and* 2 *possess a left glove, player* 3 *a right glove. A pair LR has value* 10*, singletons value* 0*. Characteristic function v:* $v(\phi) = 0$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 0$, $v(\{1, 3\}) = 10 = v(\{2, 3\}) = v(N)$.

EXAMPLE 1.2 $S \subset N$ *and* $S \neq \phi$*. S-unanimity game.* $< N, u_S >$ *or* u_S

$$u_S(T) = \begin{cases} 1, & S \subset T \\ 0, & \text{otherwise.} \end{cases}$$

Shapley proposes to divide $u_S(N) = 1$ equally among the members of S.

EXAMPLE 1.3 *$\phi \neq S \subset N$. Dual S-unanimity game u_S^**

$$u_S(T) = \begin{cases} 1, & S \cap T \neq \phi \\ 0, & \textit{otherwise.} \end{cases}$$

EXAMPLE 1.4 *Cost game $< N, c >$*
$c : 2^N \to \mathbb{R}$. $c(S)$ costs if members of S work together.

EXAMPLE 1.5 *(lady with bag) Two players are needed to transport bag to hotel. Reward 20.*
$v(\{i\}) = 0$ for $i \in N$, $v(\{i,j\}) = 20$ for $i \neq j$, $v(N) = 20$.

1.3 The Shapley value

Facts:

- The games $v : 2^N \to \mathbb{R}$ with $N = 1, 2, ..., n$ form a *linear space* of dimension $2^n - 1$. Notation G^N.
- $\{u_S | S \in 2^N \setminus \{\phi\}\}$ is a basis.

  DEFINITION 1.1 *The Shapley value $\Phi : G^N \to \mathbb{R}$ is the unique linear map with*

  $$\Phi_i(u_S) = \begin{cases} \frac{1}{|S|}, & i \in S \\ 0, & \textit{otherwise} \end{cases}$$

  for each $S \in 2^N \setminus \{\phi\}$.

- $\Phi_i(v) = \sum_{S: i \notin S} \frac{|S|!(n-1-|S|)!}{n!}(v(S \cup \{i\}) - v(S))$, convex combination. $\Phi(v) = \frac{1}{n!} \sum_\sigma m^\sigma(v)$ where $m^\sigma(v)$ is the *marginal vector* corresponding to ordering $\sigma(1), \sigma(2), \ldots, \sigma(n)$ of players.

  $$m^\sigma_{\sigma(1)}(v) = v(\sigma(1)) - v(\phi)$$

  $$m^\sigma_{\sigma(2)}(v) = v(\sigma(1), \sigma(2)) - v(\sigma(1))$$

  $$\vdots$$

  $$m^\sigma_{\sigma(n)}(v) = v(N) - v(\sigma(1), \sigma(2), \ldots, \sigma(n-1))$$

- $\Phi(u_S^*) = \Phi(u_S)$
- The unique map $\psi : G^N \to \mathbb{R}^N$ satisfying
  1. *efficiency:* $\sum_{i \in N} \psi_i(v) = v(N)$ for all $v \in G^N$
  2. *symmetry:* $\psi_i(v) = \psi_j(v)$ if $v(S \cup i) = v(S \cup j)$ for all S not containing i, and not containing j
  3. *dummy player property:* $\psi_i(v) = v(i)$ if $v(S \cup i) = v(S) + v(i)$ for all $i \notin S$
  4. *linearity:* $\psi(\alpha v + \beta \omega) = \alpha \psi(v) + \beta \psi(\omega)$. $\forall\, v, \omega \in G^N$, $\forall\, \alpha, \beta \in \mathbb{R}$

  is the Shapley value Φ.
  Other *axiomatic characterizations* by P. Young (1984), S.Hart-A.Mas-Colell (1989).
- Shapley value for cost game $< N, c >$ is

$$\Phi(c) = \sum_{S: i \notin S} \frac{|S|!(n-1-|S|)!}{n!} (c(S \cup \{i\}) - c(S)).$$

- (Airport game)
  1. small plane costs c_1
  2. larger plane costs $c_1 + c_2$
  3. largest plane costs $c_1 + c_2 + c_3$

  $< N, c >$
  $c(\phi) = 0$, $c(1) = c_1$, $c(2) = c(1,2) = c_1 + c_2$,
  $c(3) = c(1,3) = c(2,3) = c(1,2,3) = c_1 + c_2 + c_3$.

$$c = c_1 u^*_{\{1,2,3\}} + c_2 u^*_{\{2,3\}} + c_3 u^*_{\{3\}}$$

$$\Phi(c) = (\frac{1}{3}c_1, \frac{1}{3}c_1 + \frac{1}{2}c_2, \frac{1}{3}c_1 + \frac{1}{2}c_2 + c_3)$$

The users pay (equally).

1.4 Imputations-The core

For $< N, v >$ a vector $x \in \mathbb{R}^N$ is called an *imputation* if

- $x_i \geq v(i)$ for each $i \in N$ (*individual rationality*),
- $\sum_1^n x_i = v(N)$ (*efficiency*).

EXAMPLE 1.6 *The imputation set of the glove game LLR is the triangle with vertices*

$$f^1 = (10, 0, 0), f^2 = (0, 10, 0), f^3 = (0, 0, 10).$$

EXAMPLE 1.7 *The imputation set of the* 2*-person game* $< N, v >$ *with* $N = \{1, 2\}$, $v(1) = 3$, $v(2) = 4$, $v(1, 2) = 9$ *is the line segment with vertices*

$$f^1 = (5, 4), f^2 = (3, 6).$$

The Shapley value $\Phi(v)$ *is equal to* $\frac{1}{2}(f^1 + f^2)$.
Note: $f^1 = m^{(2,1)}$, $f^2 = m^{(1,2)}$.
Note that,

$$v = 3u^*_{\{1\}} + 4u^*_{\{2\}} + 2u^*_{\{1,2\}}.$$

$$\Phi(v) = 3(1, 0) + 4(0, 1) + 2(\frac{1}{2}, \frac{1}{2}) = (4, 5) = \frac{1}{2}(f^1 + f^2).$$

The imputation set of an n-person game with $v(N) > \sum_1^n v_i$ is a simplex with vertices $f^1, f^2, \ldots, f^n$ where

$$(f^k)_i = v(i), \ if \ k \neq i$$

$$(f^k)_k = v(N) - \sum_{i \in N \setminus k} (v(i)).$$

$I(v)$ contains the *core* $C(v)$ of the game.

$$C(v) = \left\{ x \in \mathbb{R}^N | \underbrace{\sum_{i \in S} x_i \geq v(S)}_{split \ of \ stability} \ \forall S \in 2^N, \underbrace{\sum_{i \in S} x_i = v(N),}_{efficiency} \right\}$$

It is a bounded polyhedral set, so a polytope.

EXAMPLE 1.8 *The core $C(v)$ of the LLR-glove game consists of one point $(0,0,10)$. $\Phi(v) = (\frac{10}{6}, \frac{10}{6}, \frac{40}{6}) \notin C(v)$.*

EXAMPLE 1.9 *The core of the 3-person unanimity game $u_{\{1,2\}}$ consists of the face conv $\{(1,0,0),(0,1,0)\}$ of $I(v)$. The Shapley value $\Phi(u_{\{1,2\}}) = (\frac{1}{2}, \frac{1}{2}, 0)$ is in the center of the core.*

EXAMPLE 1.10 *$C(v) = \phi$ for the game 'lady with the bag',*

$$\left.\begin{array}{l} x_1 + x_2 \geq 20 \\ x_1 + x_3 \geq 20 \\ x_2 + x_3 \geq 20 \end{array}\right\} \Rightarrow x_1 + x_2 + x_3 \geq 30 > v(N).$$

Bondareva(1963)-Shapley(1967) gave independently a *characterization* of games with a non-empty core.
Use is made of the *characteristic vector e^S of a coalition S*: $e^S \in \mathbb{R}^N$ and

$$e_i^S = \begin{cases} 1, & i \in S \\ 0, & otherwise. \end{cases}$$

Bondareva-Shapley: A game $< N, v >$ has a non-empty core $\Leftrightarrow < N, v >$ is a balanced game.[i.e. for each $\lambda : 2^N \setminus \{\phi\} \to \mathbb{R}_+^N$ with $\sum_{S \in 2^N \setminus \{\phi\}} \lambda_S e^S = e^N$ we have $\sum_{S \in 2^N \setminus \{\phi\}} \lambda_S v(S) \leq v(N)$].
Note that for a 3-person game with a non-empty core we have

$$\frac{1}{2}v(1,2) + \frac{1}{2}v(1,3) + \frac{1}{2}v(2,3) \leq v(1,2,3) \ (*)$$

because

$$\frac{1}{2}e^{\{1,2\}} + \frac{1}{2}e^{\{1,3\}} + \frac{1}{2}e^{\{2,3\}} = e^N$$

$$(\frac{1}{2}(1,1,0) + \frac{1}{2}(1,0,1) + \frac{1}{2}(0,1,1) = (1,1,1))$$

For the game 'lady with the bag' the condition $(*)$ is not satisfied:

$$\frac{1}{2}20 + \frac{1}{2}20 + \frac{1}{2}20 > v(1,2,3) = 20.$$

A *proof* of the 'Bondareva-Shapley' theorem can be given using a *duality theorem* from LP:
[Aumann: minimax theorem]

$$C(v) \neq \phi \Leftrightarrow v(N) = \min \left\{ e^N x | \begin{bmatrix} e^{\{1\}} \\ \vdots \\ e^S \\ \vdots \\ e^N \end{bmatrix} x \geq \begin{bmatrix} v(1) \\ \vdots \\ v(S) \\ \vdots \\ v(N) \end{bmatrix} \right\}$$

$\Updownarrow$ duality theorem
$< N, v >$ balanced $\Leftrightarrow v(N) = \max \left\{ \sum \lambda_S v(S) | \lambda_S \geq 0 \ \forall S, \lambda_S e^S = e^N \right\}$.
For $< N, v >$ and $S \subset N$ the game $< S, v >$ with players set S and $v : 2^S \to \mathbb{R}$ the restriction of v w.r.t. 2^S is called the *subgame* of $< N, v >$ corresponding to S.

Definition 1.2 *A game is called totally balanced if each subgame is balanced.*

Equivalently, a game is totally balanced, if (the game and) all subgames have a nonempty core.
Examples: linear production games, flow games, market games.
Interesting for totally balanced games are *population monotonic allocation schemes* (pmas) introduced by Sprumont (GEB 1990). They do not exist for all totally balanced games, but they exist e.g. for the subcone of convex games.

Definition 1.3 *A scheme* $[a_{S,i}]_{S \in 2^N \setminus \{\phi\}}$, $i \in S$ *is called a pmas if*

- $(a_{S,i})_{i \in S} \in C(S, v)$ *for all* $S \in 2^N \setminus \{\phi\}$, *and*
- *If* $i \in S \subset T$, *then* $a_{S,i} \leq a_{T,i}$ *(monotonicity).*

Example 1.11 *Let* $< N, v >$ *be the* 3*-person game with* $v(1) = 10$, $v(2) = 20$, $v(3) = 30$, $v(1,2) = v(1,3) = v(2,3) = 50$, $v(1,2,3) = 102$. *Then* *a pmas is the total Shapley value.*

$$\Phi(\{1,3\},v) \rightarrow \begin{array}{l} \\ N \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} \left[\begin{array}{ccc} 1 & 2 & 3 \\ 29 & 34 & 39 \\ 20 & 30 & * \\ 15 & * & 35 \\ * & 20 & 30 \\ 10 & * & * \\ * & 20 & * \\ * & * & 30 \end{array} \right].$$

$$v = 10u_1 + 20u_2 + 30u_3 + 20u_{12} + 10u_{13} + 12u_{123}.$$

This is a convex game.

1.5 Other Solutions

- The τ-*value* (Tijs 1981).
  The τ-value is the feasible compromise between the *minimum right vector* $m(v)$(disagreement point) and the *marginal vector* $M(v)$ (utopia point) for *quasibalanced games*.
  Here,

$$M_i(v) = v(N) - v(N \setminus \{i\})$$

$$m_i(v) = \max \underbrace{(v(S) - \sum_{j \in S \setminus \{i\}} M_j(v))}_{remainder for\ i \in S}$$

REMARK 1.1 *1. If $x \in C(v)$, then $m(v) \leq x \leq M(v)$.*

2. If v is convex, then $m_i(v) = v(i)$ for all $i \in N$.

3. For big boss games $\tau(v)$ is in the center of the core.

4. For total big boss games the total τ-value is a bi-mass.

- The *nucleolus*
  For each $x \in I(v)$ and $S \in 2^N \setminus \{\phi\}$

  $$e(S,x) = v(S) - \sum_{i \in S} x_i \ \textit{excess, complaint}.$$

  $\theta(x)$ is the vector of $2^n - 1$ excesses written down in decreasing order.

  $$Nu(v) = \arg\, lex \min \{\theta(x) | x \in I(v)\} \ \ \textit{nucleolus}.$$

  THEOREM 1.1 *For balanced games the nucleolus is a core element.*

- *Kernel, bargaining set* (See book of G. Owen).
- *Stable sets* or von Neumann-Morgernstern solutions (See vNM (1944), Lucas).

1.6 Bargaining games, NTU-games

- (F,d) *bargaining problem.* $F \subset \mathbb{R}^2$ *set of feasible utility pairs*(conditions on F). d *disagreement point.*
  Most well-known solutions

  - *Nash bargaining solution:*

    $$N(F,d) = \arg \max_{x \in F_d} (x_1 - d_1)(x_2 - d_2)$$

    where,

    $$F_d = \{x \in F | x_1 \geq d_1, x_2 \geq d_2\}$$

  - *Raiffa-Kalai-Smorodinsky solution:* Feasible Pareto point which is *compromise* between disagreement point d and utopia point $u = (u_1, u_2)$ where,

    $$u_i = \max \{x_i | x \in F_d\}$$

- *NTU-games* $< N, V >$
  $V(S) \in \mathbb{R}^S$ for each $S \in 2^N \setminus \{\phi\}$ Aumann-Peleg
  $Core(V)$, $SH(V)$, HARSANYI-value, NTU-value, compromise values.

1.7 Conclusions

Models: $< N, v >$, $< F, d >$, $< N, V >$
Cones of games: Balanced games, Totally balanced games, Convex games, Total big boss games, Info collecting games (Monotone veto games)
ORG's: Linear production games, Flow games, Sequencing games, Minimum spanning tree games, Holding games,...
Solution concepts: Shapley value, nucleolus, τ-value, core, pmas, bi-monotonic allocation schemes, Nash bargaining solution, RKS-solution, ...
Journals: International Journal of Game Theory (IJGT), Games and Economic Behavior (GEB), Math. Soc. Sciences (MaSS), J. Public Economic Theory (JPET), International Game Theory Review (IGTR).
Books: G. Owen, T. Driessen, I. Curiel
PhD dissertations

2 Operations Research Games

- Non-cooperative game theory $\leftrightarrow$ OR
- OR games
  - optimization
  - sharing
- Cooperative game theory $\leftrightarrow$ OR
- OR games Some historical cases (LP, MCST, flow, TS)
- Sequencing games
- Linear production situations with n owners

2.1 Non-cooperative Game Theory $\leftrightarrow$ OR

- LP $\leftrightarrow$ matrix games
  1947 visit Dantzig to von Neumann
  minimax theorems $\leftrightarrow$ *duality theorems*

- LCP $\leftrightarrow$ bi-matrix games
  Lemke Howson

$$f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$$

$$\hat{x} \perp f(\hat{x})$$

- Markov decision problems $\leftrightarrow$ Stochastic games
  L. Shapley 1953
- Control theory $\leftrightarrow$ Differential games
  R.Isaacs 1952

2.2 Operations Research Games

Optimization and Sharing

- Playground for TU-games
  - Linear production games, flow games
    G.Owen (1975), Kalai and Zemel (1982), Dubey and Shapley (1984), Granot (1986),Curiel, Derks and Tijs (1989), Reijnierse, Maschler, Potters, Tijs (1996 GEB), Gellekom, Potters, Reijnierse, Tijs and Engel (2000, GEB), Timmer, Llorca and Tijs (2000, IGTR).
  - Minimum spanning tree games
    Claus and Kleitman (1973), C. Bird (1976), Granot and Huberman (1981 – 1982), Aarts (1994), Granot, Maschler, Owen, Zhu (1996), Moretti et al (2001), Norde, Moretti and Tijs (2001), Gallekom and Potters (1999).
  - Permutation games, sequencing games
    Tijs, Parthasarathy, Potters, Rajendra Prasal (1984), Curiel, Pederzoli and Tijs (1989), Curiel, Potters, Rajendra Prasadl, Tijs, Veltman (1994,1995).
  - Travelling salesman games, delivery games
    Potters, Curiel and Tijs (1992, MP), Hamers, Borm, van de Leensel and Tijs (1999, EJOR), Granot, Hamers, Tijs (1999, MP)

    - Holding games
      Tijs, Meca and Lopez (2000)
    - Assignment, pooling
      Shapley and Shubik (1971), Potters and Tijs (1987).

- Surveys

    - 1987 Potters, Linear Optimization Games. CWI Tract 39.
    - 1997 Curiel, Cooperative Game Theory and Applications. TDLC Kluwer.
    - 2001 Borm, Hamers and Hendricks, Operation Research Games: A Survey.

2.3 Cooperative Game Theory

ORG's: Operations Research Games
Games arising from cooperation in situations where the worths $v(S)$ are obtained by solving an OR-problem.
Two problems

- OR-problem
- Sharing problem

Examples: minimum spanning tree games, fixed tree games, flow games, assignment games, transportation games, inventory games, holding games, TS-games, linear production games, sequencing games, delivery games ...

2.4 OR-games, Some historical notes

ORG's

- An optimization problem for the grand coalition
- A sharing problem

Some historical cases

1. Linear produciton games (pooling resources)

- $TOBA^n_+$
- 'Plan and pay'

$$\begin{aligned} v(N) &= \max\left\{x^Tc | x^TA \leq \sum_{i\in N} b^i, x \geq 0\right\} \\ &= \min\left\{(\sum_{i\in N} b^i)y | y \geq 0, Ay \geq 0\right\} \end{aligned}$$

$$z = (b^1\hat{y}, \ldots, b^n\hat{y}) \in C(v)$$

- Owen set $\subset$ core

2. Minimum spanning tree games (joint use of connections)

   - PMAS
   - Construct and charge

3. Max flow game

   THEOREM 2.1 *Kalai-Zemel(*1982*)*
   All max flow games have a non-empty core.

   Ford-Fulkerson algorithm can generate one core element easily.

4. Travelling salesman game
   For cost game $< N, c >$ the core is defined by

$$Core(N, c) = \left\{ x \in \mathbb{R}^N | \underbrace{\sum_{i=1}^{n} x_i = c(N)}_{efficiency}, \forall S \underbrace{\sum_{i\in S} x_i \leq c(S)}_{split\ off\ stability} \right\}$$

   REMARK 2.1 *Not all TS-games have a non-empty core. Speaker far away,universities clustered, yes.*

   Refer to: J.Potters, I.Curiel and Stef Tijs, Traveling salesman games, Math. Prog.53, 199-211, 1992.

2.5 Sequencing games

Introduction:

- Long tradition in OR: sequencing and scheduling, problems (det. and stoch.)
- Important for industrial production processes
- Important for math (NP,…,B and B,dyn.prog.,…)
- Difficult field: one decision maker (Heuristics)
- Many decision makers ('rights to be served' can be interchanged)
  - Optimal schedules
  - Cost sharing problem
- Many possible criteria

  $$\sum \alpha_i C_i(\pi), \text{ addition}$$

  $$\max C_i(\pi), \text{ due dates}$$

  …
  Ready times give extra problems.
  One machine $\sum \alpha_i C_i(\pi)$, Smith rule
  Some results with more machines
- H. Hamers dissertation.
  idea maker: Baker, Lenstra, Rinnooy Kan 1995, French

Literature Sequencing:

- I. Curiel, G. Pederzoli, S. Tijs, Sequencing games, Eur J. of Operational Research 40, 344-351, 1989.
- I. Curiel, J. Potters, V. Rajendra Prasad, S. Tijs and B. Veltman, Sequencing and cooperation, Operations Research 42, 566-568, 1994.
- I. Curiel, J. Potters, V. Rajendra Prasad, S. Tijs and B. Veltman, Cooperation in one machine, ZOR (MMOR)38, 113-129, 1993.

- 1995 PhD Dissertation Herbert Hamers.
- 1997 $\pm\,\epsilon$ TDL-C book I. Curiel
  ....

Sequencing situations and Sequencing games: 1982 Visit India, start interest in ORG's (NorthWestern 1980 KZ)

$$< N, \sigma_0, p, \alpha >$$

- The *agents* in $N = \{1, 2, \ldots, n\}$ are waiting for service.
  N: players.
- *Without cooperation* (rearrangement of position) the agents are served in the *order* $\sigma_0 : 1, 2, \ldots, n$.
  σ_0: initial order.
- The *processing time* p_i for player i is deterministic. $p = (p_1, \ldots, p_n)$: processing time vector.
  p_i: time needed to finish the job of player i.
- *Cost* for player i is $\alpha_i t$ if t is waiting time + service time.
  $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$: cost vector.

Questions:

1. What is the optimal order for N?
2. How to share the gains from cooperation?

Answers:

1. W. Smith (1956)
   Order agents with respect to urgency index: the most urgent first.

$$\alpha_1\ \alpha_2\ \alpha_3\ \ldots\ \alpha_n\ \ : \text{cost factors}$$

$$p_1\ p_2\ p_3\ \ldots\ p_n\ \ : \text{processing times}$$

$$u_1\ u_2\ u_3\ \ldots\ u_n\ \ : \text{urgency indices}$$

$$u_i = \frac{\alpha_i}{p_i}$$

EXAMPLE 2.1 $p_1 = 2, \alpha_1 = 20 \Rightarrow u_1 = 10$, $p_2 = 3, \alpha_2 = 60 \Rightarrow u_2 = 20$, $p_3 = 4, \alpha_3 = 100 \Rightarrow u_3 = 25$
Optimal order of service: 3, 2, 1.

2. CPT (1989) Go from σ_0 to optimal order.

EXAMPLE 2.2 $p_1 = 1, \alpha_1 = 20, u_1 = 20$, $p_2 = 1, \alpha_2 = 10, u_2 = 10$, $p_3 = 1, \alpha_3 = 30, u_3 = 30$
Optimal order of service: 3 1 2 *(W. Smith: optimal,*1956*).*
Obtainable from initial order by 2 *neighbour switches.*
1 2 3 $\overbrace{\Rightarrow}^{2\leftrightarrow 3}$ 1 3 2 *(gain:* 20*).*
1 3 2 $\overbrace{\Rightarrow}^{1\leftrightarrow 3}$ 3 1 2 *(gain:* 10*).*
EGS-allocation: $(*, 10, 10) + (5, *, 5) = (5, 10, 15)$ *(Switch and Share).*
This EGS-allocation lies in the core of the Sequencing game $< N, v >$ *with*

$$N = \{1, 2, 3\}, v(1) = v(2) = v(3) = v(\phi) = 0$$

$$v(1, 2) = 0 \ (\textit{1 is more urgent than 2})$$

$$v(1, 3) = 0 \ (\textit{switch not allowed because 2 in between})$$

$$v(2, 3) = 0, v(1, 2, 3) = 30 = 20 + 10$$

Note:

1. $v = 20u_{2,3} + 10u_{1,2,3}$ *convex.*

$$\begin{aligned}(5, 10, 15) &= \frac{1}{2}m^{(1,2,3)}(v) + \frac{1}{2}m^{(3,2,1)}(v) \\ &= \frac{1}{2}(0, 0, 30) + \frac{1}{2}(10, 20, 0)\end{aligned}$$

2. Sidepayments: $(25, 20, -45)$

THEOREM 2.2 • *Sequencing games are convex games.*

• $\phi(v) \in C(v)$.

- *EGS-allocation in core of game (marginals of convex games are core elements).*

$$v = 20u_{2,3} + 10u_{1,2,3}$$

$$m^{(1,2,3)} = (v(1), v(1,2) - v(1), v(N) - v(1,2)) = (0, 0, 30)$$

$$m^{(3,2,1)} = (10, 20, 0)$$

$$\phi(v) = (3\frac{1}{3}, 13\frac{1}{3}, 13\frac{1}{3})$$

$$EGS(v) = (5, 10, 15)$$

Axiomatic characterizations for the EGS-rule is available.
EGS-rule (extended to subgroups) gives a population monotonic allocation scheme

v

$$\begin{array}{l} \\ \{1,2,3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2}g_{12} + \frac{1}{2}g_{13} & \frac{1}{2}g_{12} + \frac{1}{2}g_{23} & \frac{1}{2}g_{13} + \frac{1}{2}g_{23} \\ \frac{1}{2}g_{12} & \frac{1}{2}g_{12} & * \\ 0 & * & 0 \\ * & \frac{1}{2}g_{23} & \frac{1}{2}g_{23} \\ 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$$

- In larger coalitions higher rewards
- In a convex game all core elements are pmas-extendable.

c

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{l} \{1,2,3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} &
\begin{bmatrix}
c(1) - \frac{1}{2}(g_{12} + g_{13}) & c(2) - \frac{1}{2}(g_{12} + g_{23}) & c(3) - \frac{1}{2}(g_{13} + g_{23}) \\
c(1) - \frac{1}{2}g_{12} & c(2) - \frac{1}{2}g_{12} & * \\
c(1) & * & c(3) \\
* & c(2) - \frac{1}{2}g_{23} & c(3) - \frac{1}{2}g_{23} \\
c(1) & * & * \\
* & c(2) & * \\
* & * & c(3)
\end{bmatrix}
\end{array}
$$

In larger coalitions lower costs

$$
c(S) = \sum_{i \in S} c(i) - \underbrace{\sum_{[i,j] \subset S} g_{ij}}_{v(S)}
$$

$$[\mu_{S,i}]$$

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{l} \{1,2,3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} &
\begin{bmatrix}
\alpha_1 p_1 & \alpha_2(p_1 + p_2) - g_{12} & \alpha_3(p_1 + p_2 + p_3) - g_{13} - g_{23}) \\
\alpha_1 p_1 & \alpha_2(p_1 + p_2) - g_{12} & - \\
\alpha_1 p_1 & - & \alpha_3(p_1 + p_3) - g_{13} \\
- & \alpha_2 p_2 & \alpha_3(p_2 + p_3) - g_{13} - g_{23}) \\
\alpha_1 p_1 & - & - \\
- & \alpha_2 p_2 & - \\
- & - & \alpha_3 p_3
\end{bmatrix}
\end{array}
$$

$$[EGS_{S,i}]$$

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{l} \{1,2,3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} &
\begin{bmatrix}
\alpha_1 p_1 - \frac{1}{2}g_{12} - \frac{1}{2}g_{13} & \alpha_2(p_1 + p_2) - \frac{1}{2}g_{12} - \frac{1}{2}g_{23} & \alpha_3(p_1 + p_2 + p_3) - \frac{1}{2}g_{13} - \frac{1}{2}g_{23}) \\
\alpha_1 p_1 - \frac{1}{2}g_{12} & \alpha_2(p_1 + p_2) - \frac{1}{2}g_{12} & - \\
\alpha_1 p_1 - \frac{1}{2}g_{13} & - & \alpha_3(p_1 + p_3) - \frac{1}{2}g_{13} \\
- & \alpha_2 p_2 - \frac{1}{2}g_{23} & \alpha_3(p_2 + p_3) - \frac{1}{2}g_{23}) \\
\alpha_1 p_1 & - & - \\
- & \alpha_2 p_2 & - \\
- & - & \alpha_3 p_3
\end{bmatrix}
\end{array}
$$

'Leave monotonic scheme' follows from :

$$g_{ij} = \max\{0, \alpha_j p_i - \alpha_i p_j\} \leq \alpha_j p_i$$

E.g.

$$\mu_{N,3} = \alpha_3(p_1 + p_2 + p_3) g_{13} - g_{23} \geq \alpha_3(p_1 + p_3) - g_{23} = \mu_{\{1,3\},3}$$

THEOREM 2.3 *μ is the unique stable and leave monotonic rule.*

EXERCISE 2.1 *(Permutation game)*
Let $< N, c >$ be the 3-person permutation game with cost matrix

$$K = \begin{bmatrix} 10 & 5 & 1 \\ 4 & 11 & 6 \\ 7 & 2 & 12 \end{bmatrix}.$$

- *Calculate $c(S)$ for each $S \in 2^N$*
- *Give a core element of this game.*
- *Let $< N, c_0 >$ be the zero-normalization of $< N, c >$, so*

  $$c_0(S) = c(S) - \sum_{i \in S} c(i)$$

  for each S.
  Prove that $< N, c_0 >$ is also a permutation game by giving a suitable cost matrix K_0.

EXERCISE 2.2 *(Sequencing game)*
Let the four-person sequencing situation (σ, p, α) be given by
$\sigma = (1, 2, 3, 4), p = (2, 1, 2, 1)$ *and* $\alpha = (40, 10, 60, 5)$.

- *What is the optimal order?*
- *Write the corresponding sequencing game as a linear combination of unanimity games.*
- *Calculate the EGS-allocation and the Shapley value.*

2.6 Linear Production Situations with n-owners

Agents $1, 2, \ldots, n$ own bundles of resources given by *owner ship matrix* B.

$$B = \begin{bmatrix} b_{11} & \ldots & b_{1r} \\ \vdots & & \vdots \\ b_{i1} & \ldots & b_{ir} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nr} \end{bmatrix}.$$

rows:products
columns: resources
The i-th row of the matrix B, b_i is the bundle owned by agent (player) i The columns of the matrix B named as $R_1, \ldots, R_r$ respectively are the resources. These resources can be used to make products. The possibilities are described by the *technology matrix*A.

$$A: \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_m \end{array} \begin{bmatrix} R_1 & R_2 & \ldots & R_r \\ a_{11} & a_{12} & \ldots & a_{1r} \\ a_{21} & a_{22} & \ldots & a_{2r} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \ldots & a_{mr} \end{bmatrix}$$

The first row represents the resource bundle needed for one unit of product p_1.
The market prices of the product are given by the *price vector*

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

The second row represents the *price* for one unit of p_2.
The agents in a linear production situation $< A, B, c >$ can *pool* their *resource bundles* to perform better. Also subgroups of owners can do so. **Questions:**

1. Optimal production plan

2. If all agents cooperate how to divide the reward?

$< A, B, c >$ (linear production situation)$\longrightarrow< N, v >$ (LP-game) **Question:**$v(S) =$? Is the core nonempty?
x_i make x_i units of p_i
Feasible production plans for S are vectors $x^T = (x_1, x_2, \ldots, x_m)$ with

$$x \geq 0,\ x^T A \leq \sum_{i \in S} b_i$$

$x^T A$: resources needed to execute production plan
$\sum_{i \in S} b_i$: resource bundle available for coalition S.

So

$$v(S) = \max \left\{ x^T c | x \geq 0, x^T A \leq \sum_{i \in S} b_i \right\}$$

We suppose that (we are not in heaven or)$v(N) < \infty$. Then the *duality theorem* of LP implies

$$\begin{aligned} v(N) &= \max \left\{ x^T c | x \geq 0, x^T A \leq \sum_{i \in N} b_i \right\} \\ &= \min \left\{ (\sum_{i \in N} b^i) y | y \geq 0, Ay \geq c \right\} \end{aligned}$$

y is the shadow price vector for resources.
Take $\hat{y} \in O(D_N)$, the set of optimal vectors in the dual problem for N.
Pay owner i the amount $z_i = b_i - \hat{y}$. Then $z \in C(v)$ (Owen 1975).

$$\begin{array}{cc} & \\ x_1 & \\ x_2 & \end{array} \begin{bmatrix} y_1 & y_2 & c \\ 2 & 1 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

$$\begin{array}{ccc} b^1 & 6 & 0 \\ b^2 & 0 & 5 \\ b^3 & 0 & 1 \end{array}$$

$$b^S := \sum_{i\in S} b^i.$$

Main problem

$$6x_1 + 8x_2 - \max\ !\ u.t.r.\ x \geq 0 \begin{cases} 2x_1 + x_2 \leq b_1^S \\ x_1 + 4x_2 \geq b_2^S \end{cases}$$

Dual problem

$$b_1^S y_1 + b_2^S y_2 - \min\ !\ u.t.r.\ y \geq 0 \begin{cases} 2y_1 + y_2 \geq 6 \\ y_1 + 4y_2 \geq 8 \end{cases}$$

The feasible region of the dual problem does not depend on S!

$$b^N = (6,6), \hat{y} = (2\frac{2}{7}, 1\frac{3}{7}), v(N) = 22\frac{2}{7}$$

$$x_1 = b^1\hat{y} = 6 \cdot 2\frac{2}{7} = 13\frac{5}{7}$$

$$x_2 = b^2\hat{y} = 5 \cdot 1\frac{3}{7} = 7\frac{1}{7}$$

$$x_3 = b^3\hat{y} = 1 \cdot 1\frac{3}{7} = 1\frac{3}{7}$$

$x \in C(v)$.
So we have, linear production games are totally balanced games.
i.e. all subgames have a non-empty core.
Further it is easy to find an *optimal plan* and a *core element* in one blow by looking at the primal and dual program for the grand coalition N which can be summarized by PRODUCE AND PAY.

REMARK 2.2 *The set of core elements obtained by shadow prices is called nowadays the Owen set of th LP-situation.*
"Characterization of the Owen set of linear production processes", Games and Economic Behavior, 2000, *J. van Gellekom, J.Potters, J. Reijnierse, S.Tijs, M. Engel.*

REMARK 2.3 *Linear production situations with 'veto control' over the bundles lead to all balanced games.*
"On balanced games and games with committee control", Operations Research Spectrum 11, 83 − 88, 1989, *I. Curiel, J. Derks, S.Tijs.*

REMARK 2.4 *Recently, extensions to infinite production situations (products).*

- *"Balanced games arising from infinite linear modals", Mathematical Methods of Operations Research* 50, 385 − 397, 1999,*V. Fragnelli, F. Patrone, E. Sideri, S.Tijs.*
- *"Games arising from infinite production situations", International Game Theory Review* 2, 97 − 105, 2000, *J. Timmer, N. Llorca, S.Tijs.*
- *"The Owen set and the core of semi-infinite linear production situations", SIP-conference volume*
  (Eds. M.A.López and M.A. Goberna), 2001, *S.Tijs, J. Timmer, N. Llorca, J.Sánchez-Soriano.*

Mathematical Programming Games : G.Owen 1975, Dubey-Shapley, Curiel-Derks-Tijs, Kalai-Zemel.
Deposit problems (integer programs)
Investment Problems, semi-infinite assignment / transport (integer programs, infinite programs)
Duality gaps possible: primal game $\neq$ dual game.
At first sight: Owen method does not work.
At second sight: α-core elements.
Concave programs - holding situations (Tijs-Meca-Lopez 2000)
Capital deposits (cutting problems, integer programs)
α-core (work in progress with Anja de Waegenaere and Jeroen Suijs).

3 Cases in Cooperation and Cutting the Cake

Interdisciplinary conference 'Procedural Approaches and Solutions', ZIF, August 15 − 17, 2002.

1. Introduction:
   Cases in Cooperation and Cutting the Cake, Stef Tijs, Tilburg University at present ZIF.

2. Cooperation in container transport:
   A harbour problem (trucks), Middle Rhine Problem (boats); Rotterdam
3. Telecom Problems and Game Theory:
   phoning in planes, indirect phone calls (Cambridge)
4. Concluding Remarks

3.1 Introduction

GP *II*
Invitation by e-mail: december 1999.
Tasks:

- Review of interesting historical cases and recent cases.
- Touch: What is the game practice?

GP? (provocative name)

- Kluwer TDLC 23: Preface, M. Maschler, A. Roth (GPI).
- GT $\rightleftharpoons$ Outside World

Interaction important to improve theory.
Goal: Nothing so practical as a *good theory* (Experiments, observations of behaviour,...)
Game Theory:

- Economics
- Law
- Politics
- Engineering
- Operations research
- Mathematics
- Evolutionary biology

- Social sciences
- Finance
- Environmental science

For this lecture:
$GP \equiv GT$ with an open eye outside.
Cost allocation problems $\leftrightarrow$ cooperative games:
Cooperate?
No/ Yes?
If yes, with whom? In which form? In which intensity? How to allocate cost savings?
Model: Cooperative game ($< N, c >$) $\rightarrow$ Solution concepts $\rightarrow$ Advice: allocation proposal $\underbrace{\leftrightarrow}_{interaction}$ customers $\underbrace{\leftrightarrow}_{interaction}$ Model: Cooperative game.
N: players, $S \subset N$: potential coalition, $c(S)$:costs for coalition S in cooperation.
Workshop Game Theory and Engineering: Politecnico, Milano, June 9, 2005.
Stef Tijs, Dima Genova, CentER, Tilburg.
Cases in Cooperation and Cutting the Cake, CentER DP, October 2004−108, Stef Tijs and Rodica Branzei, to appear in 'Procedural Approaches in Conflict Resolution', (Ed. M. Raith), Springer Verlag, 2007.
Outline:

1. Introduction
2. Classical Cases (The past?)
3. Cooperation in container transport
4. Telecom problems and game theory
5. Concluding remarks

Gap Between Theory and Practice?
Supply side: sophisticated models solutions
Demand side: Transparant advice solutions
Game Practice Conferences $(1998, 2000, 2002(2x))$
ZIF Bielefeld (Oct. 2001-Aug. 2002) Interdisciplinar

Procedural Approaches to Conflict Resolution
Experimental Game Theory (Vernon Smith)
DILEMMAS:
Theoretical insights - classical solutions $\longleftrightarrow$ Easy transparant advices - To the spot solutions
The Past?
In the next scheme some 'historical cases'.
On the left side indication of case and a relevant question.
on the right side game theoretical notions related to the case.

1. Babylonian Talmud (Kethuboth, Fol. 932): nucleolus
   Who suffers what from bankruptcy? consistency

2. The Tennessee Valley : τ-value
   What costs electricity...? core

3. The Birmingham airport: Shapley value
   What to pay for a landing?

4. Cornell Univ. telephone bills: Aumann-Shapley pricing
   How much to pay for long distance calls?

5. Xerox: Serial cost sharing
   What pay the different users of one facility?

6. Dutch Telecom, British Telecom:
   Cones K_2 K_3
   $\phi = nu = \tau, \phi = nu$ (merge)
   How to split gains in 'phoning in planes' 'indirect phone calls'?

7. Container transport: big boss games sequence of TU-game
   What to pay the owner of routing program?

8. Montana farmers: Weighted Shapley values, Constrained egalitarian rules
   What to contribute to maintainance of irrigation system?

9. Italian railways project: Generalized airport games
   How to share railways infra structure costs?

10. ORG's (Operation Research Games): Tailor made solutions, Subclasses of games
    Optimize for N and share?

Objectives of GT-group TU

Nijmegen (Dept. of Math.) (master thesis) $\stackrel{1991}{\rightarrow}$ Tilburg (Dept. of Econometrics and OR) (stage)
Peter Borm - Anne van den Nouweland - Gert Jan Otten - Stef Tijs

Firms approached us

- Objectives
    - Contribution to final phase of students
    - Gain respect from economic colleagues
    - Learn advising
- Resulted in
    - new insights (transparancy, axioms)
    - new GT results
    - Papers

MAIN QUESTION

How to cut the cake if it comes to cooperation?
CAKE: cost savings, extra gains
M.Shubik in the fifties: Cooperative game theory can be helpful
Cooperative game: $< N, v >$, $N = \{1, 2, \ldots, n\}$: *players set*, participants
$v : 2^N \rightarrow \mathbb{R}$: *worth function*
$v(S)$: value (worth) of coalition S.
$v(N)$: value of the grand coalition.
Solution concepts: Shapley value, τ-value, nucleolus, core, ...

EXAMPLE 3.1 $N = \{1, 2\}, v(\{1\}) = 10, v(\{2\}) = 15, v(\{1, 2\}) = 65.$
Extra cake $65 - (10 + 15) = 40$, *equal split* 20 *and* 20.
$(10 + 20, 15 + 20) = \phi(v) = Nu(v) = \tau(v) \in C(v)$.

problems (Cases):

1. Firms in Rotterdam harbour: Container transport by trucks to Germany
2. Middle Rhine region: Container transport by ships
3. Phoning in planes

3.2 Cooperation in container transport

1. A HARBOUR PROBLEM (very preliminary)
   Advising was 'succesfull'.
   empty trucks
   Involved: 3 firms (1, 2 and 3) who wanted to cooperate in truck transport. *Advantages:* Extra gains

   - good planning
   - less (partially empty) trucks

   Problem 1: 1 had developed excellent routing program.
   Question (Quarrel): What have 2, 3 to pay for the developing costs?
   We: proposal which gave only indirect answer to the question.
   Model: Sequence of TU-games which reflects the unequality in equipment.

$$\begin{array}{lccccccc} t= & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ & v & v & v & w & w & w & \\ or & v & v & w & w & w & w & \dots \end{array}$$

   $v(2,3) < w(2,3)$ (Big boss games). After two, three years 2 and 3 can have also a good program.
   Then,

$$\begin{array}{cccccc} \phi(v) & \phi(v) & \phi(v) & \phi(w) & \phi(w) & \dots \\ Nu(v) & Nu(v) & Nu(v) & Nu(w) & Nu(w) & \\ \tau(v) & \tau(v) & & & & \end{array}$$

   Result: After a month they cooperated on this basis. (Average of ϕ, Nu, τ! Barbarians!) *Later:* After two years split off $\{1\}, \{2,3\}$.
   Homework:(still to be done) Sequences of cooperative games.

2. MIDDLE RHINE PROBLEM
   Observation of GHR (Rotterdam Harbour Office) No cooperation in container transport on Middle Rhine; cooperation on upper Rhine or lower Rhine.
   GHR $\longleftrightarrow$ GTTU
   Same problem as with trucks. Here boats.
   Study of problem: $< N, v >$, ... , report.
   report $\longrightarrow$ possible cooperators: Passive behavior.
   Q: Why disappeared our report in the drawers? (No success)
   A: ?

   - Different cultures of firms? (Royal, five year old,...)
   - Indirect advising via GHR?

   Literature:'Goed, beter, binnenvaart' (Game theoretic research for possibilities of cooperation in container transport on the river Rhine), Angeline Nielen, April 1994.

3.3 Telecom Problems and Game Theory

In this section rough treatment of 2 problems
Problem 2: **Reward sharing** in applying **' in direct phone calls '**. Operators can serve more customers in busy hours using the 'sleeping part' of the world (Cambridge group).
France $\stackrel{direct\ call}{\rightarrow}$ USA $\leftarrow$ Japan
indirect call from France to USA.
Problem 3: **Reward sharing** and cost sharing in **'phoning in planes'**. Calling in plane to home e.g. via ground station. **Cooperation of many countries important.**

Literature:

- R.Gibbens, F.Kelly, G.Kope and M. Whitehead, "Coalitions in the international network", in A. Jensen and V. Iversen (Editors), Teletraffic and Datatraffic in a Period of Change, ITC 13, $93-98$, 1991. (Indirect phone calls)
- A van den Nouweland, P. Borm, W. van Golstein Brouwers, R. Groot Bruinderink and S. Tijs, "A Game Theoretic Approach to Problems in

Telecomunication", Management Science, 42, 1996, 294 – 303. (Phoning in planes)

Leads to special cones of TU-games: *k-games*, where some solutions coincide and are in the core.
Crucial role: 'basis of unanimity games.'

- Indirect phone calls
  Carriers in the countries are the *players*. *Profit* generated through a rerouting of a call:
  3 international carriers involved

  1. carrier in original country
  2. a transit carrier
  3. carrier in destination country

$$v(S) = \sum_{T\subset S,\ |T|=3} v(T) \quad (*) \quad \forall S$$

  In study of Gibbens et al.
  3 zones

  – American
  – European
  – Asian

  From $(*)$

$$v = \sum_{T:|T|=3} v(T)u_T.$$

  So $v \in K_3$ which implies

  – Core is non-empty
  – $\Phi(v) = \tau(v) \in C(v)$

  K_3: cone generated by unanimity games u_T

$$u_T(S) = \begin{cases} 1, & T \subset S \\ 0, & otherwise. \end{cases}$$

- Phoning in planes TFTS (*a terrestrial flight telephone system*) is a public telephone service for pessengers in airplanes in which the telephone connections are established by *radio communication to a near ground station*, from where the connections are provided to the destination subscriber using the *existing network*.
  Need for
  - Ground stations
  - Communications apparatus in planes

  Q: How to share costs and rewards?

  Rejected proposals:

  1. gains proportional to investment costs of ground stations (Germany planned a lot of ground stations) Each operator keeps the revenues of their ground station. Protests (not stable).

  Our proposal: based on solution of a cooperative game.
  Cooperative (phoning) game $< N, v >$. Players are the *countries* $1, 2, \ldots 12$.

  1. Netherlands (PTT and KL)
  2. Germany (Bundespost and LH)
  3. etc.. (Iberia, Spanish telecom)

  Costs in countries paid by country. (c_i).
  Gains a_{ij} (12×12): planes of country j above country i.

$$\bar{a}_i = a_{ii} - c_i.$$

  Gives a matrix of size $12x12$.

$$\begin{bmatrix} \bar{a}_1 & a_{12} & a_{13} & \bar{a}_{1,12} \\ a_{21} & \bar{a}_2 & \ldots & \\ \vdots & & \vdots & \\ & & a_{ij} & \\ \vdots & & \vdots & \\ a_{12,1} & & & a_{12} \end{bmatrix}.$$

rows:ground station
columns: planes.

$$v(S) = \underbrace{\sum_{i,j \in S,\ i \neq j} a_{ij}}_{v_2(S)} + \underbrace{\sum_{i \in S} \bar{a}_i}_{v_1(S)}$$

Claim: $v = v_1 + v_2$ where v_1 is *additive* game and $v_2 \in K_2$.

$$v_2 = \sum_{i,j \in N,\ i \neq j} a_{ij} u_{\{i,j\}}$$

a_{ij}'s *determined by Deutsche Bundespost* using traffic intensities, capacities and load factors.
Proposal:

$$x_i = \overbrace{\bar{a}_i}^{home} + \overbrace{\frac{1}{2}\sum_{j \neq i} a_{ij}}^{planes\ above\ i} + \overbrace{\frac{1}{2}\sum_{j \neq i} a_{ji}}^{planes\ of\ i\ elsewhere}$$

(= ($\frac{1}{2}$ of row sum of row i + column sum of column i of matrix))
Then

1. $x \in C(v)$ and v is convex.

2. $x = \Phi(v) = \tau(v) = Nu(v)$

$$v \in K_1 + K_2.$$

Proof:

1. $\bar{a} + \Phi(v) \in C(v)$. An additive game is convex and the sum of convex games is convex. So v is convex.

2. Since $v = v_1 + v_2$ with v_1 additive and $v_2 \in K_2$ we obtain
$\Phi(v_2) = \tau(v_2) = Nu(v_2)$ and since
$f(v_1 + v_2) = \bar{a} + f(v_2)$ if $f \in \{\phi, \tau, Nu\}$ we obtain 2.

3.4 Concluding Remarks

1. Know each other
   advisor $\longleftrightarrow$ clients.
   For six months, econometrics student full time, PhD-student one day a week, weekly meetings in Tilburg, seminar halfway.

2. Know the problem thoroughly.

3. The influence of the way of interaction.

4. Success?

5. Publication rights problem

6. Spin off

Q: The influence of the way of interaction
Advisors $\longleftrightarrow$ Potential group of cooperators.
Is there an ideal form of interaction?
Are some ways (uniformly) better than others?
Our experience

- Harbour problem. Interaction with problem owner.

- Phoning in planes.

- Middle Rhine. Indirect advising. Harbour office, city Rotterdam.
  GTTU $\longleftrightarrow$ GHK.

No experience with

- Group advising

- Internal advising

Our experience: A 'problem owner' in the group of cooperators is an interesting method.
Success?

- Follow up often secret. Problem owner should be content; cooperation starts

- The researcher should be happy (Ethics, scientific impact).

Problem: Publication right!
Spin off:

- Advising experience
  Way of interaction, transparant solutions, decomposition, solutions on subclasses, 'Hide' advanced game theory: 'bounded rational solutions' for 'b.r.' clients.
- Game practice
- Italian Railways
- Toolbox
  - $K_2, K_3, \ldots, K_p = cone\,\{u_S|\,|S| = p\}$
    Adiitivity τ, merge properties ϕ
  - Need for dynamic cooperative game theory
  - Development of 'bounded rational' solutions
  - Ad hoc solutions

PROBLEM
Editor management science want real data. PTT research (KPN now) want secrecy (also after two years).
Authors $\stackrel{think}{\rightarrow}{}^{3x}$ COMPROMISE
Condensation of data: Benelux $\{B, N, L\}$ became one player Scandinavia ... etc..
Is this acceptable ...? Does our rule, applied on condensation, g,ve a 'related' reward distribution?
Is the rule 'merge proof'?
Jean Derks and Stef Tijs, On merge properties of the Shapley values, International Game Theory Review, 2, 249 − 257, 2000.

4 The First Steps with Alexia, the average lexicographic value (CentER DP $2005-123$)

Outline

1. Introduction (Game Theory, NE $\longleftrightarrow$ Core, classes of games)
2. The average lexicographic value
3. AL for convex games, big boss games, ...
4. Properties of AL
5.
   - Exact games and exactifications
   - The exactification of a big boss game
6. Partially defined games
7. Further research
8. Summary

Beatrix $\leftrightarrow$ W.A.$\leftrightarrow$ MAXIMA
Maxima

- AMalia
  average **m**arginal vectors, Future Queen (no core business), Shapley value, Weber set.
- ALexia
  average **l**exicographic vectors, core business, Lexcore.

4.1 Introduction

Game Theory

- Math. Models of Conflict and Cooperation
- Toolbox for social sciences

Game Theory

- Non-cooperative Game Theory
  - Dominant solution concepts: Nash Equilibrium
  - Stability: J.F. Nash (1950), A. Cournot (1838)
  - Existence: J.F. Nash (fixed point theorems)

– Approximate solutions
– Selections and refinements
– Axiomatizations

- Cooperative Game Theory
  – Dominant solution concepts: Core
  – Stability: D.B. Gillies (1953), F.Y. Edgeworth (1881)
  – Existence: Bondareva (duality)
  – Approximate solutions
  – Selections and refinements
  – Axiomatizations

*TODAY:*A new core selection for balanced games

REMARK 4.1 *The famous Shapley value is not a core selection.*

Classical Notions:

- Cooperative game: $< N, v >$ or v.
  $N = \{1, 2, \ldots, n\}$: set of *players*
  2^N: subsets of N (*coalitions*)

$$v : 2^N \to \mathbb{R},\ v(\phi) = 0$$

  $v(S)$ *worth* (value) of coalition S.

- Core of game $< N, v >$: $C(v)$

$$C(v) = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | \underbrace{\sum_{i \in S} x_i \geq v(S)\ for\ each S\ \in 2^N}_{stability\ conditions\ or\ coal.\ rationality\ cond.},\ \underbrace{\sum_{i \in S} x_i = v(N)}_{efficiency\ condition}, \right\}$$

  polytope

- Balanced Games: games with a non-empty core

EXAMPLE 4.1 *Convex games* $\subset$ *Balanced games*
$< N, v >$ *is convex if*

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \ for\ all\ S \subset T \subset N \setminus \{i\}\,.$$

EXAMPLE 4.2 *Big boss games*
$< N, v >$ *is a (total)* big boss game *with* n *as big boss if*

1. *big boss property:* $v(S) = 0$ *if* $n \notin S$.

2. *monotonicity property:*

$$S \subset T \Rightarrow v(S) \leq v(T)$$

3. *concavity property:*

$$n \in S \subset T \subset N \setminus \{i\} \Rightarrow v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T).$$

τ*-value, nucleolus.*

4.2 The AL-value

Alexia:

- for games
- for MOP

May 11, 2005 Barcelona (Princess: June 26, 2005)
Q: Looking for interesting core selections

- Nucleolus (Schmeidler)
- Per capita nucleolus (Grotte)
- Core center (Gonzalez-Diaz and Sánchez-Rodriguez)
  ⋮
- Shapley value for convex games (Shapley (1971))

1. Lexicographic optimization(s) play(s) a role.
   History (Debreu:utility functions do not exist; Schmeidler:nucleolus, Multiobjective programming)

2. *Averaging* plays a role to avoid discrimination.

$$\phi(v) = \frac{1}{n!} \sum_{\sigma} m^{\sigma}(v) \ (Shapley)$$

3. Also interesting for MOP.

Lexicographic maxima and the lexicographic center

- Let $\sigma = \sigma(1), \sigma(2), \ldots, \sigma(n)$ be an *ordering* of $1, 2, \ldots, n$. Then $\geq_\sigma$ is the linear ordering on $\mathbb{R}^N$ defined by

$$x \geq_\sigma \ if \begin{cases} x = y \ or \ x_{\sigma(1)} > y_{\sigma(1)} \ or \\ x_{\sigma(1)} = y_{\sigma(1)} \ and \ x_{\sigma(2)} > y_{\sigma(2)} \ or \\ \vdots \\ or \\ x_{\sigma(k)} = y_{\sigma(k)} \ for \ k \in \{1, 2, \ldots, n-1\} \ and \ x_{\sigma(n)} > y_{\sigma(n)} \end{cases}$$

  for all $x, y \in \mathbb{R}^{\mathbb{N}}$.

- Let X be a compact, convex, non-empty subset of $\mathbb{R}^{\mathbb{N}}$ (e.g. $I(v), C(v), W(v)$). Then the *σ-lexicographic maximum* (the unique largest element of X w.r.t. the ordering $\geq_\sigma$ (σ selector)) is denoted by $S^\sigma(X)$.

- The *lexicographic center* of X:

$$\frac{1}{n!} \sum_{\sigma \in \Pi(N)} S^\sigma(X)$$

  is denoted by $AL(X)$.

- For balanced games $AL(C(v))$ is a core selection denoted by $AL(v)$.

4.3 AL for convex games, big boss games, ...

EXAMPLE 4.3 $< N, v >, N = \{1, 2, 3\}, v(1) = v(2) = v(3) = 0,$
$v(1, 2) = \ldots, v(1, 3) = \ldots, v(2, 3) = \ldots, v(1, 2, 3) = 9.$

$S^{(3,1,2)}(I(v)) = S^{(3,2,1)}(I(v)) = (0, 0, 9),$
$S^{(2,1,3)}(I(v)) = S^{(2,3,1)}(I(v)) = (0, 9, 0),$
$S^{(1,2,3)}(I(v)) = S^{(1,3,2)}(I(v)) = (9, 0, 0).$

So $AL(I(v)) = \frac{1}{6}((9, 0, 0) + \ldots + (0, 0, 9)) = (3, 3, 3) = CIS(v).$
CIS*: Center of imputation set.*

EXAMPLE 4.4 *(Convex game)*
$v(i) = 0, v(1, 2, 3) = 9, v(i, j) = 2$ *for all* $i, j \in N, i \neq j.$
$S^{(1,2,3)}(C(v)) = m^{(3,2,1)}(v) = (7, 2, 0),\ S^{(2,1,3)}(C(v)) = m^{(3,1,2)}(v) = (2, 7, 0)$

$$\begin{aligned} C(v) &= conv\,\{(7, 2, 0), \ldots, (7, 0, 2)\} \\ &= conv\,\{m^{\sigma}(v) | \sigma \in \Pi(N)\} \\ &= conv\,\{S^{\sigma}(C(v))\}\,. \end{aligned}$$

($AL(C(v)) =$*)* $AL(v) = \frac{1}{n!} \sum S^{\sigma}(C(v)) = \phi(v).$

THEOREM 4.1 *For convex games,* $v : AL(v) = \phi(v).$

EXAMPLE 4.5 *(Big boss game)*
$N = \{1, 2, 3\}, v(1) = v(2) = v(3) = 0, v(1, 2, 3) = 9,$
$v(1, 2) = 0, v(1, 3) = 6, v(2, 3) = 5$
$\tau(v) = (2, 1\frac{1}{2}), 5\frac{1}{2}$
$S^{(3,1,2)}(C(v)) = S^{(3,2,1)}(C(v)) = (0, 0, 9)$ *(big boss point B),*
$S^{(2,1,3)}(C(v)) = S^{(1,2,3)}(C(v)) = (4, 3, 2)$ *(union point U),*
$S^{(1,3,2)}(C(v)) = (4, 0, 5), S^{(2,3,1)}(C(v)) = (0, 3, 6).$

$$C(v) = conv\,\{(0, 0, 9), (4, 3, 2), (4, 0, 5), (0, 3, 6)\}$$

$$AL(v) = \frac{1}{6} \sum_{\sigma} S^{\sigma}(C(v)) = (2, 1\frac{1}{2}, 5\frac{1}{2}) = \tau(v).$$

THEOREM 4.2 *For big boss games the lexicographic value coincides with the* τ*-value.*

4.4 Properties of Alexia

1. $AL_i(v) = v(i), \forall i \in N$ Individual Rationality.

2. $\sum_{i=1}^{n} AL_i(v) = v(N)$ Efficiency.

3. $AL(v) \in C(v)$ Core Selection.

4. $AL(kv + a) = kAL(v) + a$ S-equivalence.

5. $AL_i(v) = AL_j(v)$ if i, j *symmetric* players
   $(i, j \notin S \Rightarrow v(S \cup i) = v(S \cup j))$ SYM.

6. $AL_i(v) = v(i)$ if i is a *dummy* player i.e.
   $\forall i \notin S \, [v(S \cup \{i\}) = v(S) + v(i)]$ DUM.

7. • $C(v) = C(w) \neq \phi \Rightarrow AL(v) = AL(w)$.
   • $(\phi \neq C(v) = C(v^E)) \Rightarrow AL(v) = AL(v^E)$

   INVEX

8. E-additivity

   $AL(v + w)^E = AL(v^E) + AL(w^E) \; if \; C(v^E + w^E) = C(v^E) + C(w^E)$.

   ...

Open Problem: Axiomatic characterization of AL.

4.5

- Exact games and exactifications

  Definition 4.1 *A game $< N, v >$ is called an exact game if for each $S \in 2^N \setminus \{\phi\}$, there is an element $x^S \in C(v)$ such that*

  $$\sum_{i \in S} x_i^S = v(S)$$

  The *cone* of exact games is denoted by EX^N
  (restricted)

  $$ADD^E : AL(v + w) = AL(v) + AL(w)$$

  for each $v, w \in EX^N$ with $C(v + w) = C(v) + C(w)$.

– $EX^N \subset BA^N$
– $BA^N \to EX^N$
$v \mapsto v^E$, where v^E is the exactifications of v.

$$v^E(S) = \min \left\{ \sum_{i \in S} x_i | x \in C(v) \right\}$$

Interpretation: $v^E(S)$ is an 'update' of $v(S)$ if N is 'core-oriented'.

REMARK 4.2 *Convex games are exact game (but not the way around).*

EXAMPLE 4.6 $N = \{1, 2, 3\}$
player 1*: strong buyer* 120*, player* 2*: weak buyer* 80*, player* 3*: seller.*

$S =$	(1)	(2)	(3)	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$
$v(S) =$	0	0	0	0	120	80	120

$$C(v) = conv \{(0, 0, 120), (40, 0, 80)\}$$

$S =$	(1)	(2)	(3)	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$
$v^E(S) =$	0	0	80	0	120	80	120

By shifting hyperplane $x_3 + v(3)$ to $x_3 = v^E(3)$ the hyperplane becomes a supporting hyperplane of the core.

$$v^E(S) is\ convex : v^E = 80u_3 + 40u_{1,3}$$

Since 2-person and 3-person games which are exact are also convex we obtain

THEOREM 4.3 *For 2-person and 3-person balanced games $< N, v >$ we have*

$$AL(v) = \phi(v^E)$$

Definition 4.2 *If*

$$\phi \neq C(v) = \underbrace{I(v)}_{imputation\ set} = \left\{x \in \mathbb{R}^N | x(N) = v(N), x_i \geq v(i)\ for\ each\ i \in N\right\}$$

then $< N, v >$ *is called a simplex game.*

Theorem 4.4 *For a simplex game* $< N, v >$

$$AL(v) = \underbrace{CIS(v)}_{center\ of\ imputation\ set} = \phi(v^E)$$

Theorem 4.5 *For a dual simplex game* $< N, v >$, *(where*

$$C(v) = I^*(v) = \left\{x \in \mathbb{R}^N | \sum_{i=1}^{n} x_i = v(N), x_i \leq v(N) - v(N \setminus i), \forall i \in N\right\})$$

we have

$$AL(v) = ESNR(v) = \tau(v) = Nu(v) = \phi(v^E)$$

Example 4.7 *ϵ-DERKS/KUIPERS (A 4-person game which is exact and not convex,* $AL(v) \neq \phi(v^E)$*)*
Let $\epsilon \in (0, 1]$ *and* $< N, v >$ *be a game with* $N = \{1, 2, 3, 4\}$

$$v(S) = \begin{cases} 7\ if\ \ |S| = 2 \\ 12\ if\ \ |S| = 3 \\ 22\ if\ \ |S| = 4 \end{cases}$$

$v(1) = \epsilon, v(2) = v(3) = v(4) = 0$. $< N, v >$ *is not a convex game because*

$$v(123) - v(12) = 5 < v(1, 3) - v(1) = 7 - \epsilon$$

The core has 24 *extreme points:*

- 12 *extreme points which are permutations of* $x = (10, 5, 5, 2)$.
- 9 *extreme points which are permutations of* $(7, 7, 8, 0)$ *but with first coordinate which is not equal to zero.*

- $(\epsilon, 7-\epsilon, 7-\epsilon, 8-\epsilon), (\epsilon, 7-\epsilon, 8-\epsilon, 7-\epsilon), (\epsilon, 8-\epsilon, 7-\epsilon, 7-\epsilon)$.

This implies that

- $< N, v >$ *is exact.*
- $S^{\sigma}(v)$ *is a permutation of* $(10, 5, 5, 2)$.
- $AL(v) = (5\frac{1}{2}, 5\frac{1}{2}, 5\frac{1}{2}, 5\frac{1}{2}) \neq \phi(v^E) = \phi(v) = (5\frac{1}{2} + \frac{1}{4}\epsilon, 5\frac{1}{2} - \frac{1}{12}\epsilon, 5\frac{1}{2} - \frac{1}{12\epsilon}, 5\frac{1}{2} - \frac{1}{12}\epsilon)$.

- The exactification of a big boss game

1. The *core* of a big boss game $< N, v >$ (parallelotope):
$$C(v) = \left\{ x \in \mathbb{R}^n | 0 \leq x_i \leq M_i(v) \forall i \in N \setminus n, \sum_{i=1}^{n} x_i = v(N) \right\}$$
$$M_i(v) = v(N) - v(N \setminus i)$$
2. The *extreme points of the core* $\left\{ p^T | T \subset N \setminus \{n\} \right\}$ with $i \in N \setminus n$
$$p_i^T = \begin{cases} M_i(v) \ if \ i \in T \\ 0 \ if \ i \in N \setminus T \cup \{n\} \end{cases}$$
$$p_n^T = v(N) - \sum_{i \in T} M_i(v).$$
3. $S^{\sigma}(v) = p^{T(\sigma)}$ where $T(\sigma) = \{i \in N \setminus \{n\} \,| \sigma(i) < \sigma(n)\}$.
$$AL(v) = (\frac{1}{2}M_1(v), \frac{1}{2}M_2(v), \ldots, \frac{1}{2}M_{n-1}(v), v(N) - \frac{1}{2}\sum_{i=1}^{n-1} M_i(v)) = \tau(v) = Nu(v).$$
4. v^E is a convex game because
$$v^E = \sum_{i \in N \setminus \{n\}} M_i(v) u_{\{i,n\}} + (v(N) - \sum_{i=1}^{n-1} M_i(v)) u_{\{n\}}$$
5. $\phi(v^E) \underbrace{=}_{v^E \ convex} AL(v^E) \underbrace{=}_{INVEX} = AL(v) = \tau(v)$.

   EXAMPLE 4.8 $N = \{1, 2, 3\}, v(S) = 0 \ if \, 3 \notin S, v(1,3) = 70, v(2,3) = 60, v(1,2,3) = 100$.
   $M_1(v) = 40, M_2(v) = 30, AL(v) = \tau(v) = (20, 15, 65)$.
   $U = p^{\{1,2\}} = (40, 30, 30), B = p^{\phi} = (0, 0, 100)$

4.6 Partially defined games

Patrone-Pusillo-Torre-Caprari-Tijs

$\mathbb{F} \subset 2^N$: set of feasible coalitions. $N \in \mathbb{F}, \ldots$
$v : \mathbb{F} \to \mathbb{R}$ is called $\mathbb{F}$-balanced if

$$\phi \neq C_{\mathbb{F}}(v) := \left\{ x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \ for\ all\ S \in \mathbb{F} \right\}$$

and $C_{\mathbb{F}}(v)$ bounded.
$v : \mathbb{F} \to \mathbb{R} \Rightarrow \bar{v} : 2^N \to \mathbb{R}$ extension of v where

$$\bar{v}(S) = \min \left\{ \sum_{i \in S} x_i | x \in C_{\mathbb{F}}(v) \right\}$$

ψ on G^N a classical solution $\bar{\psi}(N, \mathbb{F}, v) = \psi(N, v)$.
Study: $\bar{AL}$ in these situations.

4.7 Further Research

- Monotonicity properties of AL.
- Continuity property of AL.
- Consistency properties of AL.
- Axiomatizations
- Numerical aspects
- Cones with perfect kernel systems and AL.
- Relations with other core selections.
- Extensions of AL from $BA^N \to \mathbb{R}^N$ to $G^N \to \mathbb{R}^N$.
- More relations with other solution concepts.

4.8 Summary

$$AL(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} S^{\sigma}(v)$$

average of lexicographic maxima

$$INVEX : AL(v^E) = AL(v)$$

For games $< N, v >$ where $< N, v^E >$ is convex we have $AL(v) = \phi(v^E)$.
E.g. simplex games, dual simplex games, convex games, big boss games, ...
Further research: Axiomatization of AL.

5 Bankruptcy Problems and Games

Outline

1. Literature and the Talmudic examples
2. Bankruptcy problems and bankruptcy rules
3. Bankruptcy games (Game theoretic and other solutions)
4. Compact sets $\leftrightarrow$ Cooperative games
5. Hydraulic Rationing
6. Concluding remarks
7. Summary

5.1 Literature

- B. O'Neill, A problem of rights arbitration from the Talmud, MaSS 2, 345-371, 1982.
- R. Aumann and M.Maschler, Game Theoretic Analysis of a Bankruptcy Problem from the Talmud. JET, 36, $195 - 213$, 1985

- I.Curiel, M.Maschler, S.Tijs, Bankruptcy Games, ZOR 31, 143 – 159, 1987.
- M.Kaminsky, Hydraulic Rationing, Mass 40, 131 – 155, 2000.
- W.Thomson, Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: A survey, MaSS 45, 249 – 297, 2003.

5.2 Bankruptcy problems and bankruptcy rules

- Bankruptcy problem: $< \underbrace{E}_{estate}; d_1, \underbrace{d_2}_{claim\ of\ player\ 2}, \ldots, d_n >$

$$0 \leq E \leq \sum_{i=1}^{n} d_i,\ 0 \leq d_1 \leq d_2 \leq \ldots \leq d_n.$$

- Collection of n-person bankruptcy problems: BR^N.
- Opportunity set of BR-problem (E, d).

$$D(E,d) = \left\{ x \in \mathbb{R}^N | 0 \leq x_i \leq d_i\ \forall i \in N, \sum_{i=1}^{n} x_i = E \right\}$$

- Bankruptcy rule: $F : (E, d) \mapsto F(E, d) \in D(E, d)$
- Examples
  - Proportional rule:

$$PROP(E,d) = \alpha d \textit{ with } \alpha \textit{ such that } \alpha \sum_{i=1}^{n} d_i = E.$$

  - Constrained equal award rule

$$CEL(E,d) = (d_1 \wedge \alpha, d_2 \wedge \alpha, \ldots, d_n \wedge \alpha)$$

    where $\alpha \in \mathbb{R}_+$ such that $\sum_{i=1}^{n} d_i \wedge \alpha = E$.

– Constrained equal loss rule:

$$CEL(E,d) = ((d_1 - \alpha)_+, (d_2 - \alpha)_2, \ldots, (d_n - \alpha)_+)$$

where for $x \in \mathbb{R} : x_+ = \max\{x, 0\}$, and β such that

$$\sum_{i=1}^{n} (d_i - \beta)_+ = E.$$

– Run to the bank rule (O'Neill)

$$AR = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} R^\sigma$$

Example 5.1 $RUN(E,d)$,(200; 100, 200, 300)

	1	2	3
R^{123}	100	100	0
R^{132}	100	0	100
R^{213}	0	200	0
R^{231}	0	200	0
R^{312}	0	0	200
R^{321}	0	0	200
	200	500	500

$$AR = 33\frac{1}{3}, 83\frac{1}{3}, 83\frac{1}{3}$$

– Adjusted proportional rule (Curiel-Mashler-Tijs)

$$\begin{aligned} APROP(500; 100, 200, 300) &= (0, 100, 200) + PROP(200; 100, 100, 100) \\ &= (66\frac{2}{3}, 166\frac{2}{3}, 266\frac{2}{3}). \end{aligned}$$

$$m_i = (E - d(N \setminus i))_+$$

5.3 Bankruptcy games

$< N, v_{E,d} >$, $< N, v^*_{E,d} >$

$$v_{E,d}(S) = \max\{0, E - d(N \setminus S)\} \text{ (pessimism)}$$

$$v^*_{E,d}(S) = \min\{d(S), E\} \text{ (optimism)}$$

THEOREM 5.1 *$RUN(E,d) = \phi(v^{E,d}) = AL(v^{E,d})$.*

THEOREM 5.2 *$APROP(E,d) = \tau(v_{E,d)})$.*

THEOREM 5.3 *$TAL(E,d) = Nu(v_{E,d)})$.*

THEOREM 5.4 *1. $< N, v^{E,d} >$ is a convex game.*

2. $< N, v^{E,d}_* >$ is a concave game.

$$\begin{aligned}
(v_{E,d})^*(S) &= v^{E,d}(N) - v^{E,d}(N \setminus S) \\
&= E - \max\{0, E - d(N \setminus N \setminus S)\} \\
&= \min\{E - 0, E - (E - d(S))\} \\
&= \min\{E, d(S)\} \\
&= v^{E,d}_*(S).
\end{aligned}$$

So, the dual of the pessimistic game $v^{E,d}$ is the optimistic game $v^{E,d}_*$. The pessimistic game is convex, so the optimistic game is concave.

$$RUN(E,d) = \phi(v^{E,d}) = AL(v^{E,d}).$$

EXAMPLE 5.2 $(E,d) = (500; 100, 200, 300)$

$S =$	(1)	(2)	(3)	(1,2)	(1,3)	(2,3)	(1,2,3)
$v_{E,d}(S) =$	0	100	200	200	300	400	500
$v^*_{E,d}(S) =$	100	200	300	300	400	500	500

$$C(v) = conv\{(0,200,300), (100,200,200), (100,100,300)\}.$$

RUN

	1	2	3	
L^{123}	100	200	200	m^{321}
L^{132}	100	100	300	m^{231}
L^{213}	100	200	200	m^{312}
L^{231}	0	200	300	m^{132}
L^{312}	100	100	300	m^{213}
L^{321}	0	200	300	m^{123}
	400	1000	1600	

The marginal vectors are in the core. So, $< N, v_{E,d} >$ is a convex game.

$$\phi(v) = AL(v) = RUN(E,d) = (66\frac{4}{6}, 166\frac{4}{6}, 266\frac{4}{6}).$$

THEOREM 5.5 *Given (E,d) the core of $v_{E,d}$ and the opportunity set coincide:*

$$C(v_{E,d}) = D(E,d)(= \left\{ x \in \mathbb{R}^n | 0 \leq x \leq d, \sum_{i=1}^{n} x_i = E \right\})$$

proof:

1. $D(E,d) \subset C(v_{E,d})$
   Let $x \in D(E,d)$. Then
   - $\sum_{i=1}^{n} x_i = E = v^{E,d}(N) (= \max\{0, E - d(N \setminus N)\})$.
   - $\sum_{i \in S} x_i = E - \sum_{i \in N \setminus S} x_i \geq E - \sum_{i \in N \setminus S} d_i$, $\sum_{i \in S} x_i \geq 0$
     $\forall S$, $\sum_{i \in S} x_i \geq v_{E,d}(S)$.So, $x \in C(v_{E,d})$.
2. $C(v_{E,d}) \subset D(E,d)$
   Take $x \in C(v_{E,d})$. Then $\sum_{i=1}^{n} x_i = v^{E,d}(N) = E$.
   $\forall i \in N, x_i \geq v^{E,d}(\{i\}) = \max\{0, E - d(N \setminus N \setminus \{i\})\} \geq 0$.

$$\begin{aligned} x_i &\leq v^{E,d}(N) - v^{E,d}(N \setminus \{i\}) \\ &= E - \max\{0, E - d_i\} \\ &= \min\{E, d_i\} \\ &\leq d_i. \end{aligned}$$

   So, $x \in D(E,d)$.

5.4 Compact sets $\leftrightarrow$ Cooperative games

$$D \subset \mathbb{R}^n \text{ compact} \leftrightarrow \underbrace{< N, v_1^D >}_{minimum\ right\ game\ c.t.D}, \underbrace{< N, v_2^D >}_{utopia\ game\ c.t.D}.$$

$N = \{1, 2, \dots, n\}$. Take coalition $S \subset D$. Then

$$v_1^D(S) = \min \left\{ \sum_{i \in S} x_i | x \in D \right\}$$

$$v_2^D(S) = \max \left\{ \sum_{i \in S} x_i | x \in D \right\}$$

Proposition 5.1 *Suppose $D \subset \mathbb{R}^n$ compact and $\exists \alpha \in \mathbb{R}, \forall x \in D$.* $[\sum_{i=1}^n x_i = \alpha]$.
Then $v_2^D = (v_1^D)^*$.

Proof:

$$\begin{aligned} v_2^D(S) &= \max \left\{ \sum_{i \in S} x_i | x \in D \right\} \\ &= \max \left\{ \alpha - \sum_{i \in N \setminus S} x_i | x \in D \right\} \\ &= \alpha - \min \left\{ \sum_{i \in N \setminus S} x_i | x \in D \right\} \\ &= v_1^D(N) - v_1^D(N \setminus S) \\ &:= (v_1^D)^*(S). \end{aligned}$$

Q: v_1^D, v_2^D for $D = D(E, d)$?

Theorem 5.6 *Let (E, d) be a bankruptcy problem and let*

$$D = \left\{ x \in \mathbb{R}^n | 0 \le x \le d, \sum_{i=1}^n x_i = E \right\}.$$

Then $< N, v_{E,d} > = < N, v_1^D >$ and $< N, v_{E,d}^* > = < N, v_2^D >$.

Proof sketch: Because of the 'duality results' we need only to prove that $v^*_{E,d} = v^D_2$.

$$v^*_{E,d}(N) = E = v^D_2(N).$$

Take $S \in 2^N$. (Run to the bank where players in S first).
If $\sum_{i\in S} d_i \leq E$, take $x^S \in D$ with $x_i = d_i$ for each $i \in S$. Then

$$v^D_2(S) = \sum_{i\in S} d_i = \min\{E, d(S)\} = v^*_{E,d}(S).$$

If $\sum_{i\in S} d_i > E$, take $x^S \in D$ such that $x_i = d_i$ for the first l,
$x_{s_{l+1}} = E - \sum_{r=1}^{l} d_{s_r}$.

$$S = \left\{ \underbrace{s_1}_{d_1} < s_2 < \underbrace{s_3}_{d_3} < \underbrace{\ldots}_{E-d_1-d_2-d_3} < s_k \right\}.$$

Then $\sum_{i\in S} d_i = E = \max\{E, d(S)\} = v^*_{E,d}(S)$.

$$v^D_1 : \text{exact game, for } n = 2,3 \; v^D_1 : \text{ convex.}$$

$$v^D_2 : \text{dual exact game.}$$

DEFINITION 5.1 *A game* $< N, v >$ *is exact if* $\forall S, \exists x \in C(v)$ $[\sum_{i\in S} x_i = v(S)]$.

DEFINITION 5.2 *A game* $< N, v >$ *is dual exact if* $\forall S, \exists x \in C^*(v)$ $[x(S) = v(S)]$.

DEFINITION 5.3

$$C^*(v) = \left\{ x \in \mathbb{R}^N | \sum_{i=1}^{n} x_i = v(N), x(S) \leq v^*(S), \forall S \right\}.$$

$< N, a, d, E >$

$$D = \left\{ x \in \mathbb{R}^n | 0 \leq a_i \leq x_i \leq d_i \; \forall i \in N, \sum_{i=1}^{n} x_i = E \right\}.$$

Generalized bankruptcy with claims d_i and minimum rights. Leads to v^D_1 exact, v^D_2 dual exact.

5.5 Hydraulic rationing

(Kaminski, 2000)
$(E, d) = (400; 100, 200, 300)$

1. $CEA(400; 100, 200, 300) = (100, 150, 150)$.
2. $PROP(400; 100, 200, 300) = (66\frac{4}{6}, 133\frac{2}{6}, 200)$.
3. $CEL(500; 100, 200, 300) = (100, 200, 300) - (33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$.

EXERCISE 5.1

$$TAL(E, d) = \begin{cases} CEA(E, \frac{1}{2}d) \; if \; \frac{1}{2}\sum_{i=1}^{n} d_i \geq E \\ d - CEL(D - E, \frac{1}{2}d) \; if \; \frac{1}{2}\sum_{i=1}^{n} d_i \leq E \end{cases}$$

Make a hydraulic system (Solution in Kaminsky).

5.6 Concluding Remarks

- BR-literature influences the taxation litrature (P. Young).

  $$< E, (d_1, d_2, \ldots, d_n) > \leftrightarrow < \underbrace{T}_{tax\ to\ be\ collected}, (i_1, \underbrace{i_2}_{income\ of\ player\ 2}, \ldots, i_n) >$$

  $0 \leq T \leq \sum_{k=1}^{n} i_k$.

  $$O(T, i) = \left\{ x \in \mathbb{R}^n | 0 \leq x_k \leq i_k \; \forall k \in N, \sum_{k=1}^{n} x_k = T \right\}$$

- Axiomatic approach
  Many BR-rules are characterized with a list of properties (Thomson survey).
- Non-cooperative approach (Dagan-Voly)
- Application
  "Two approaches to the problem of sharing delay costs in joint projects", Ann. of O.R. 109, $359 - 374$, 2002 (Branzei-Ferrari- Fragnelli-Tijs).

Bankruptcy problem
$(E,d) = (E, d_1, d_2, \ldots, d_n)$, $\sum_{i=1}^n d_i \geq E, d_i \geq 0$
E is *capital* left, d_i is *claimed capital* by client i. **Q:** How to divide E among the claimants?
Many solutions (answers)

$$PROP_i(E,d) = \frac{E}{\sum_{i=1}^n d_i} d_i$$

$$CEA(E,d) = (d_1 \wedge \alpha, \ldots, d_n \wedge \alpha) \ where \ \alpha \ni \sum_{i=1}^n (d_i \wedge \alpha) = E.$$

$$CEL(E,d),\ TAL(nu),\ RUN(\phi),\ AP(\tau \ of \ < N, v_{E,d} >).$$

Fish catch reduction (Kim Hang Pham Do)
$(Q, r_1, r_2, \ldots, r_n)$, $Q \leq \sum_{i=1}^n (r_i,\ r_i \geq 0$.
Q: amount of fish to be caught this year, r_i: catch (right) of agent i last year.
Similar to bankruptcy problem. Leads to rules $PROP(Q,r), CEA(Q,r), \ldots, AP(Q,r)$ and games $v_{Q,r}, v^*_{Q,r}$.

5.7 Summary

- $\underbrace{< E, d >}_{BR-problems} \longmapsto \underbrace{v_{E,d},\ v^*_{E,d}}_{dual\ pair\ of\ games}$ $v_{E,d}$ convex, $v^*_{E,d}$ concave.
- BR-rules: $RUN(\phi)$, CEA, CEL, $TAL(Nu)$, $PROP$, $APROP, (\tau)$
  Truncation property of a rule F

  $$F(E, d_1, d_2, \ldots, d_n) = F(E, d_1 \wedge E, \ldots, d_n \wedge E).$$

  A game theoretic solution exists $\Leftrightarrow$ F has truncation property (Curiel-Maschler-Tijs).
- Compact sets $D \leftrightarrow$ Cooperative games v_1^D, v_2^D
  minimum right game, utopia game.
  For $v^{E,d} : C(v^{E,d}) = D(E,d)\ v_1^{D(E,d)} = v^{E,d}$
- Hyraulic systems and bankruptcy solutions

6 Cooperative Games and Auctions

Scheme

1. Introduction
2. Single object auctions (complete information among bidders),More identical object auctions.
3. Total big boss games (MVC)
   market game, auction game (bi-monotonic allocation scheme)
4. Convex games
   peer group game, ring game (pmas)
5. Ring games $\leftrightarrow$ Auction games
6. Summary, Further research
7. References

In preparation: Cooperative games arising from deterministic auction situations, R.Branzei-V. Fragnelli-A. Meca-S.Tijs.
Related earlier work:

- Benefit sharing in holding situations (S.Tijs-A.Meca-M.Lopez) EJOR $162, 251-269, 2005$.
- Information collecting situations and bi-monotonic allocation schemes (R.Branzei-S.Tijs-J.Timmer) MMOR $54, 303-313, 2001$.
- On big boss games (S.Muto-M.Nakayama-J.Potters-S.Tijs) The Econ.Studies Quarterly, $39, 303-321, 1988$.
- Tree-connected peer group situations and peer group games (R.Branzei-V.Fragnelli-S.Tijs), MMOR $55, 93-106, 2002$.

6.1 Introduction

A *cooperative game* is a pair $< N, v >$ where $N = \{1, 2, 3, \ldots, n\}$ (set of players), $v : 2^N \to \mathbb{R}$ and $v(\phi) = 0$ (characteristic function on set of coalitions).

Cooperative games:

- Strategic games
- Economic Situations
  - Exchange markets
  - Oligopoly
  - Auctions
  - Info collecting situations
  - Bankruptcy situations
- OR-situations
  - assignment
  - min. cost spannig trees
  - sequencing
  - linear production
  - holding
  - flow
- Solution concepts
  Core

$$C(v) = \left\{ x \in \mathbb{R}^N | \underbrace{\sum_{i=1}^{n} x_i = v(N),}_{efficiency} \underbrace{\sum_{i \in S} x_i \geq v(S),}_{stability} \forall S \in 2^N \right\}$$

  Shapley value

$$\phi(v) = \sum_{S \in 2^N} \alpha_S \frac{e^S}{|S|} \; if \; v = \sum_{S \in 2^N} \alpha_S u_S$$

where

$$u_S(t) = \begin{cases} 1, & S \subset T \\ 0, & otherwise. \end{cases}$$

($u_S(T)$:S-based unanimity game).
τ**-value**, $\tau(v)$: feasible compromise between marginal vector and minimum right vector.

- Classes of games
  balanced games, totally balanced games, convex games, peer group games, total big boss games,...

Two corners in a market situation
n possible buyers and 1 seller
$N = \{1, 2, \ldots, n\}$, $N' = N \cup \{n+1\}$.Market game $< N', v >$

- Seller?
  How to approach buyers (price setting, bargaining, auction,...)? If auction, which kind of auction? How to attract many possible buyers? How to avoid collusion (ring formation) among a set of buyers? How to act in case of collusion?

- Buyers?
  How to cooperate (secretly)? How to bid? How to divide the gains? Buyers ring game $< N, r >$.

- Game theorist? What kind of game is a market game? Auction result in core? What kind of game is a ring game? Dependent on auction type (GET)? Division rules? Characterization. Which solution concepts are interesting?

Recent Books

- V.Krishna (2002) Auction Theory, San Diego, CA: Academic Press.

- P.R.Milgrom (2004) Putting Auction Theory to work, Cambridge, UK: Cambridge University Press.

- Paul Klemperer (2004) Auctions: Theory and Practice, Princeton and Oxford: Princeton University Press.

- Chapter1: A survey of auction theory
  Standard auction types, revenue equivalence.
- Chapter2: Why every economist should learn some auction theory
  Strong connections: auction theory $\leftrightarrow$ standard economic theory.
- Chapter3: What really matters in auction design
  preventing collusive, predatory and entry-deterring behavior.
- Chapter4: Using and abusing auction theory
- Chapter5 $-$ 8: Case study auctions, The $3G$ Mobile-phone.

6.2 Single object auctions (complete information)

- References (Books): Krishna (2002), Milgrom(2004),...
- Old history: Babylon (500BC), 193A.D.: Roman Empire sold by Praetorian Guard by means of an auction (the winner Didius Julianus was beheaded two months later).
- All kind of objects sold in auctions: fish, flowers, paintings, antiques, long term securities, rights to use electromagnetic spectrum.
- Classical auction types
  - Open ascending price or English auction: price increased till there is only one bidder left.
  - Open descending price auction or Dutch auction: price lowered till one agent interested.
  - Sealed-bid first-price auction: bidders submit bids in sealed envelopes highest bidder obtains object and pays his bid (lottery).
  - Sealed-bid second price auction (Vickrey auction): Bidders submit bids in sealed envelope highest bidder obtains object and pays second highest bid.

EXAMPLE 6.1 *Agents* $1, 2, 3$ *are interested in object owned (and auctioned)by agent* 4. *Value (private) of object for agent* $i \in \{1, 2, 3\} : w_i$
$w_1 = 170, w_2 = 90, w_3 = 50$. *We suppose that* w_1, w_2, w_3 *known among buyers.*

- *Dutch auction: price lowers; if lowered till* $90+\epsilon$ *player*1 *shows interest, gets the object and pays* $90+\epsilon$*.*
  Reward allocation $\approx (80-\epsilon, 0, 0, 90+\epsilon)$.

- *English auction: Player* 1 *stays (bidding) till others become silent.*
  Reward allocation $\approx (80, 90, 0, 90)$.

- *First price: Player* 1 *hands in envelope with bid* $90+\epsilon$ *and others ...(?)*

- *Second price: Player* 1 *hands in envelope with bid* w_1*. Truth telling is here a weakly dominant strategy.*

Single object auction: $< w_1, w_2, w_3; 4:1 >$.
Auction outcome:$(w_1 - w_2, 0, 0, w_2)$ is extreme point of core of market game $< N, v >$ where $N = \{1,2,3,4\}$, $v(\{i\}) = 0$ for $i \in \{1,2,3,4\}$,
$v(S) = 0$ if $4 \notin S$, $v(1,4) = v(1,2,4) = v(N) = w_1 = v(1,3,4)$,
$v(2,4) = v(2,3,4) = w_2$, $v(3,4) = w_3$.
If $w_1 = w_2$, then $U = B$, $C(v) = \{0,0,0,w_2\}$.

REMARK 6.1 *(vNM* 1944*: weak buyer* 1*, strong buyer* 2*, seller* 3*).*$w_1, w_2, 0$ *value of object by* $1,2,3$ *and* 3 *owns.*

Game $< N', v >$, $N' = \{1,2,3\}$, $v(i) = 0$ for all i.
$v(1,2) = 0, v(1,3) = w_1, v(2,3) = w_2, v(1,2,3) = w_1 . U = (w_1 - w_2, 0, w_2)$ is the auction outcome. $B = (0, 0, w_1)$.
Auction with two identical objects: Players (bidders) want only one object.
$< w_1, w_2, w_3; 4:2 >$
Auction outcome: $(w_1 - w_3, w_2 - w_3, 0, 2w_3) = U$

- $w_1 \geq w_2 > w_3$
  $B = (0,0,0,w_1+w_2), U = (w_1 - w_3, w_2 - w_3, 0, 2w_3)$

- $w_1 > w_2 = w_3$
  $U = (w_1 - w_3, 0, 0, 2w_3)$

- $w_1 = w_2 = w_3$
  $B = U = (0,0,0,2w_3)$

- English auction: 3 leaves at w_3.

- Dutch auction: $1, 2$ put button at w_3^+.
- Vickrey: Truth w_3 3 th price
- Sealed bid where bid is paid if: $b_1 = w_3^+, b_2 = w_3^+, b_1 \leq w_3$ winner.
- Sealed bid, 3th+4th price.

$< w_1, w_2, w_3; 4 : 3 >$, $U = (w_1, w_2, w_3, 0)$
$< w_1, w_2; 3 : 2 >$, $U = (w_1, w_2, 0)$, $B = (0, 0, w_1 + w_2)$.

6.3 Total big boss games

Let $N = \{1, 2, \ldots, n\}$. A game $< N \cup \{n+1\}, v >$ is a total big boss game ($v \in TOBA_{n+1}^N$) if the following 3 properties hold.

1. Big boss property: for each $S \subset \{1, 2, \ldots, n\}, v(S) = 0$.
2. Monotonicity property: $S \subset T \Rightarrow v(S) \leq v(T)$.
3. Concavity on $\mathbb{B}$ property: for $S, T \in \mathbb{B}$ and $j \in N$ such that $S \subset T$, $j \notin T$:

   $$v(S \cup \{j\}) - v(S) \geq v(T \cup \{j\}) - v(T)$$

   where $\mathbb{B} := \left\{S \in 2^{N \cup \{n+1\}} | n+1 \in S\right\}$ is the set of bigg boss coalitions.

Example 6.2 4-*person (holding) game which is a total big boss game with player* 4 *as bigg boss.*

- $v(S) = 0$ *if* $4 \notin S$ *(big boss property).*
- $9 = v(3, 4) \leq 13 = v(2, 3, 4)$ *etc..(monotonicity).*
- $9 = v(3, 4) - v(4) \geq 7 = v(2, 3, 4) - v(2, 4)$,
  $5 = v(1, 3, 4) - v(3, 4) \geq 2 = v(1, 2, 3, 4) - v(2, 3, 4)$ *etc..*

Theorem 6.1 *(Tijs-Meca-Lopez, EJOR* 2005*) Holding games are total big boss games.*

EXAMPLE 6.3 *3-person subgame with player set* $\{1,2,4\}$ *is also a total big boss game:* $N=\{1,2,4\}$.

$$v(1)=v(2)=v(4)=v(1,2)=0, v(1,4)=6, v(2,4)=6, v(1,2,4)=11.$$

$$M_1(v)=11-6=5, M_2(v)=v(1,2,4)-v(1,4)=5.$$

$$U=(M_1(v),M_2(v),v(N)-M_1(v)-M_2(v))=(5,5,1), B=(0,0,11),$$

where U *is the union point and* $B=(0,0,11)$ *is the big boss point.*

$$C(v)=\left\{x\in\mathbb{R}^{\{1,2,4\}}|\sum_{i\in N}x_i=11, 0\le x_1\le M_1(v)=5, 0\le x_2\le M_2(v)=5\right\}$$

papalellogram.
$\tau(v)=\frac{1}{2}(B+U)=\frac{1}{2}(0,0,11)+\frac{1}{2}(5,5,1)=(2\frac{1}{2},2\frac{1}{2},6)$. $\tau(v)$ *is the average of big boss point and union point. It is the bary center of the core.*
The extended τ*-value is a bi-monotonic allocation scheme:*

$$\begin{array}{l} \\ (1,2,4) \\ (1,4) \\ (2,4) \\ (4) \end{array}\begin{bmatrix} 1 & 2 & 4 \\ 2\frac{1}{2} & 2\frac{1}{2} & 6 \\ 3 & * & 3 \\ * & 3 & 3 \\ * & * & 0 \end{bmatrix}\begin{array}{l} \\ 11 \\ 6 \\ 6 \\ 0 \end{array}$$

In larger coalition player 4 *is better off, other players worse off.*

THEOREM 6.2 *Let* $N=\{1,2,\dots,n+1\}$ *and let* $<N,v>$ *a total big boss game with* $n+1$ *as big boss. Let* $B=(0,0,\dots,0,v(N)), U=(M_1(v),\dots,M_n(v)), v(N)-\sum_{i=1}^{n}M_i(v)$. *Then*

1. $$C(v)=\left\{x\in\mathbb{R}^{n+1}|\sum_{i=1}^{n+1}x_i=v(N), 0\le x_i\le M_i(v), \forall i\in\{1,2,\dots,n\}\right\}.$$

   is a paralellotope.

2. $\tau(v)=\frac{1}{2}(B+U)\in C(v)$.

3. *Each core element is bi-mas extendable.*

4. *τ on the class of total big boss games is a bi mar (bi-monotonic allocation rule).*

5. *τ is additive on $BB(N, n+1)$.*

EXAMPLE 6.4 *(vNM:weak buyer (player2), strong buyer (player1), seller (player3))*
Player3 has an object (for him of value 0). The value of the object is for player1: w_1, for player2: w_2. $0 < w_2 < w_1$. $B = (0, 0, w_1), U = (w_1 - w_2, 0, w_2)$.
Game $< N, v >$: $N = \{1, 2, 3\}$. $v(i) = 0$ for all $i \in N$. $v(1,2) = 0, v(1,3) = w_1, v(2,3) = w_2, v(1,2,3) = w_1$. Suppose we use an auction (first price sealed bid, second price sealed bid, English, Dutch). Then outcome is U.

EXAMPLE 6.5 $(170, 90, 50; 4:1)$
$v(\phi) = v(1) = v(2) = v(3) = v(4) = v(12) = v(13) = v(2,3) = v(1,2,3) = 0, v(1,4) = 170, v(2,4) = 90, v(3,4) = 50, v(1,2,4) = 170, v(1,3,4) = 170, v(2,3,4) = 90, v(2,3,4) = 170.$

$$v(S \cup \{i\}) = \max\{w_i | i \in S\}.$$

Big boss property: $4 \notin S \Rightarrow v(S) = 0$. Let us denote for each total big boss game $< S, v >$ where $n+1 \in S$ the union point by u^S.
Then u^S is the reward vector generated by the classical auction procedures in $< (w_i)_{i \in S \setminus \{n+1\}}; n+1 : 1 >$.
Further $\left[u_i^S\right]_{S \in \mathbb{B}, i \in S}$ is a bi-mon. alloc. scheme. In our example,

$$\begin{array}{l} \\ 1234 \\ 124 \\ 134 \\ 234 \\ 14 \\ 24 \\ 34 \\ 4 \end{array} \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 80 & 0 & 0 & 90 \\ 80 & 0 & \bullet & 90 \\ 120 & \bullet & 0 & 50 \\ \bullet & 40 & 0 & 50 \\ 170 & \bullet & \bullet & 0 \\ \bullet & 90 & \bullet & 0 \\ \bullet & \bullet & 50 & 0 \\ \bullet & \bullet & \bullet & 0 \end{array}\right] \begin{array}{l} \\ 170 \\ 170 \\ 170 \\ 90 \\ 170 \\ 90 \\ 50 \\ 0 \end{array}$$

The bi-mass expresses the facts that if more potential buyers show up in the auction the better (the worse) for the seller (the already present buyers).

Claim: Let $< w_1, w_2, \ldots, w_n; n+1 : k >$ a market situation, where $n+1$ owns $k \leq n$ objects.
Let $< N \cup \{n+1\}, v >$ be the corresponding market game with

$$v(S) = 0 \; if \; n+1 \notin S.$$

$$v(S) = \sum_{i \in S \setminus \{n+1\}} w_i \; if \; n+1 \in S \; and \; |S| \leq k+1.$$

$$v(S) = \max \left\{ \sum_{i \in U} w_i | U \subset S \setminus \{n+1\}, |U| = k \right\} \; if \; n+1 \in S \; and \; |S| > k+1.$$

Then

- $< N', v >$ is a total big boss game with big boss $n+1$.
- $dim(C(v)) = d \leq k$, $dimC(v) = k \Leftrightarrow w_k > w_{k+1}$.
- $|ext(C(v))| = 2^{dim(C(v))}$.

6.4 Convex games (Shapley, 1971)

Definition 6.1 *A game $< N, v >$ is called a convex game if for all $S, T \in 2^N, i \in N$ with $S \subset T \subset N \;\; \{i\}$ we have*

$$v(S \cup \{i\} - v(S) \leq v(T \cup \{i\}) - v(T).$$

Increasing marginal return property.

- The convex games form a *cone* in the game space: if $< N, v_1 >, < N, v_2 >$ are convex games with $\lambda_1 \geq 0, \lambda_2 \geq 0$, then $< N, \lambda_1 v_1 + \lambda_2 v_2 >$ is a convex game.

  Example 6.6 *$< N, u_S >$ is a convex game for each $S \in 2^N \setminus \{\phi\}$.*

  Example 6.7 *$< N, \sum_{k=1}^{m} \alpha_k u_{S_k} >$ is convex if $\alpha_k \geq 0$ for all k.*

- Let $(w_1, w_2, \ldots, w_n; n+1 : 1)$ be the *auction situation*, where agent $n+1$ auctions 1 object and where value of the object is w_i for buyer i $(i \in \{1, 2, \ldots, n\})$ and $w_1 > w_2 > \ldots > w_n$. Then the ring game $< N, r >$ corresponding to this auction situation we define by

  $$r(S) = \begin{cases} 0, & 1 \notin S \\ w_1 - w_{k+1}, & [1,k] \subset S, [1,k+1] \textit{ is not in } S. \end{cases}$$

  for $k \in \{1, 2, \ldots, n\}$.
  for each $S \in 2^N$.[Other name: peer group game. See $[2, 5]$].

(Here $[1,k] := \{1, 2, \ldots, k-1, k\}$). Cooperating groups of buyers are called rings in the literature.

Example 6.8 $(170, 90, 50; 4 : 1)$. *$< N, r >$ is given by $N = \{1, 2, 3\}$, $r(\phi) = 0$.*
$r(1) = 80, r(1,2) = 120, r(1,2,3) = 170, r(2) = r(3) = r(2,3) = 0, r(1,3) = r(1) = 80.$
Note that $r(S)$ is the gain a coalition S of buyers can obtain in cooperating in the bidding.
E.g. *if $S = \{1, 2\}$ in a sealed bid second price auction by bidding $b_1 = 170, b_2 = 0$ together with $b_3 = 50$ the object goes to player1 for price 50. Without cooperation $(b_1, b_2, b_3) = (170, 90, 50)$: player 1 obtains the object for price 90. Gain $90 - 50 = 40$. Note that*

$$r = (170 - 90)u_{\{1\}} + (90 - 50)u_{[1,2]} + (50 - 0)u_{[1,3]}$$

Then

$$r = (w_1 - w_2)u_{[1]} + (w_2 - w_3)u_{[1,2]} + (w_3 - w_4)u_{[1,3]}$$

where $w_4 := 0$.
So $< N, r >$ is a convex game.

Q: How for other auction formats?
A: A sort of equivalence theorem.

Theorem 6.3 *The ring game $< N, r >$ corresponding to the auction situation $(w_1, w_2, \ldots, w_n; n+1 : 1)$ is a convex game.*

$$r = \sum_{k=1}^{r} (w_k - w_{k+1}) u_{[1,k]}, \ w_{n+1} := 0.$$

Collusions:
$r(1) = w_1 - w_2, r(1,2) = w_1 - w_3, r(1,2,3) = w_1 - w_4, r(1,2,4,5) = w_1 - w_3.$

$$r = \sum_{i=1}^{r} (w_i - w_{i+1}) u_{[1,i]}.$$

$$\tau(r) = (w_1 - w_2)e^1 + PROP(w_2, (w_2, w_2, w_3, w_4, w_5)).$$

- Graham-Marschall-Richard AER 1990: Different payments within a bidder coalition and the Shapley value.
- Branzei-Fragnelli-Tijs, MMOR 2002: Tree connected peer group games.

The τ-value for ring games: For $w \in W^N = \{x \in \mathbb{R}^N | x_1 \geq x_2 \geq \ldots \geq x_n\}$.
Let $< N, r^w >$ be the ring game corresponding to market situation $< w_1, w_2, \ldots, w_n; n+1 : 1 >$.
Let $f^\tau : W^N \to \mathbb{R}^N$ be defined by $f^\tau(w) = \tau(r^w)$.Then f^τ satisfies:

- Efficiency:

$$\sum_{i \in N} f_i^\tau(w_1, w_2, \ldots, w_n) = w_i (= r^w(N)).$$

- First right proportionality:

$$f^\tau(w_1, w_2, \ldots, w_n) = (w_1 - w_2, 0, \ldots, 0) + f^\tau(w_2, w_2, w_3, \ldots, n)$$

  for all $w \in W^N$.

- Proportionality:

$$f^\tau(w_2, w_2, w_3, \ldots, w_n) = \alpha(w_2, w_2, w_3, \ldots, w_n).$$

Let ψ be a division rule for $\{r^w | w \in W^N\}$. Take $f^\psi(w) = \psi(r^w)$. Then

Theorem 6.4 *(Characterization of the τ-value) ψ is the τ-value iff f^ψ satisfies EFF, FIRST RIGHT, PROP.*

Note that r^w is a convex game.
In van den Brink [6] there is a characterization for the Shapley value for ring games.
The Shapley value for ring games For 3 players:

$$\phi(r^w) = (\frac{w_3}{3}, \frac{w_3}{3}, \frac{w_3}{3}) + (\frac{w_2 - w_3}{2}, \frac{w_2 - w_3}{2}, 0) + (w_1 - w_2, 0, 0).$$

F is efficient iff $F = \phi$.

- Tail dependence:

  $$F_k(w_1, w_2, \ldots, w_k, w_{k+1}, \ldots, w_n) = F_k(w_1', w_2', \ldots, w_{k-1}', w_k, w_{k+1}, \ldots, w_n), \forall k.$$

- Symmetry: If $w_i = w_j$, then

  $$F_i(w_1, w_2, \ldots, w_n) = F_j(w_1, w_2, \ldots, w_n).$$

- Efficiency:

  $$\sum_{k=1}^{n} F_k(w_1, w_2, \ldots, w_n) = w_1.$$

- R. van den Brink (2004): Null or zero players, TI $2064 - 127$.
- D.Graham, R.Marschall, J.F. Richard (1990): Differential payments within a bidder coalition and the Shapley value, AER 80, $493 - 510$.

6.5 Ring games $\leftrightarrow$ Auction games

N convex, $N \cup \{n+1\}$ total big boss. To discover the interesting relation we consider for $< w_1, w_2, w_3; 4:1 >=< 170, 90, 50, 4:1 >$ the auction game v, the dual auction game v^* and the ring game τ.

$$v^*(S) = v(N') - v(N' \setminus S).$$

Note that

$$r = 80u_{[1]} + 40u_{[1,2]} + 50u_{[1,2,3]}.$$

(A ring has a value 0 if 1 is not in the ring). Further $v^*|_2 \{1,2,3\} = \tau$.
Duality result: $< w_1, w_2, \ldots, w_n; n+1 : 1 >$

$$v^*(S) = r(S) \ for \ S \in 2^N.$$

$$v(N^{'}) = r(S) + v(N^{'} \setminus S).$$

$$v(N^{'}) - v(N^{'} \setminus S) = r(S).$$

$$v^*(S) = r(S).$$

6.6 Summary, Further research

- Summary
  Single object auction $< w_1, w_2, \ldots, w_n; n+1 : 1 >$
  - Auction game (market game) $< N \cup \{n+1\}, v >$
    Total big boss game possesing bi-mass, union point in core special role.
  - Ring game $< N, r >$
    convex (peer group game) possesing pmas

  Relation between games: $r = v^*|_{2^N}$
- Further research
  - relations bi-mass (auction game) $\leftrightarrow$ pmas (ring game)
  - multi object auctions $< w_1, w_2, \ldots, w_n; n+1 : k >$ and cooperative games.
  - Auctions with incomplete information $\leftrightarrow$ cooperative games incomplete information.

Auctions and Cooperative games (Summary):
Main notions:

1. Market situation: $< w_1, w_2, \ldots, w_n; n+1 : 1 >$, $w_{n+1} := 0$.

2. Market game (auction game): $< N \cup \{n+1\}, v >$ where $N = \{1, 2, \ldots, n\}$ (demanders) and $v(S) = 0$ if $n+1 \notin S$, $v(S \cup \{n+1\}) = \max\{w_i | i \in S\}$.

3. Buyers ring game: $< N, r >$ with $r(S) = 0$ if $1 \notin S$

$$r(S) = w_1 - w_{k+1} \ if \ [1, k] \subset S, k+1 \notin S.$$

Main facts:

1. Outcome equivalence theorem: the four classical auctions lead in the market game to the same payoff allocation: $(w_1 - w_2, 0, \ldots, 0, w_2)$.

2. The market game is a total big boss game with the seller as bigg boss. The core is 0 or 1-dimensional: $C(v) = conv\,\{w^1 e^{n+1}, (w^1 - w^2)e^1 + w_2 e^{n+1}\}$.

3. The ring game (peer group game) is a special convex game

$$r = \sum_{k=1}^{n} (w_k - w_{k+1}) u_{[1,k]}.$$

$$\phi_i(r) = \sum_{k=1}^{i} \frac{w_k - w_{k+1}}{k}.$$

$$\tau(r) = (1-\alpha)((w^1 - w^2), 0, 0, \ldots, 0) + \alpha(w_2, w_2, w_3, \ldots, w_n).$$

4. Duality relation: $v^*|_{2^N} = r$.

6.7 References

[1] R. Branzei, V. Fragnelli, S. Tijs, *On the computation of the nucleolus of line graph peer group games*,Scientific Annals of the Alexandra Ioan Cuza University, Computer Science Section, 9, 79-91 (2000).

[2] R. Branzei, V. Fragnelli, S. Tijs, *Tree-connected peer group situations and peer group games*, MMOR, 55, 93-106, (2002).

[3] R. Branzei, V. Fragnelli, A. Meca, S. Tijs , *On cooperative games arising from deterministic auction situations with complete information among bidders*, (2005: in preparation).

[4] R. Branzei, T. Solynosi, S. Tijs, *Strongly essential coalitions and the nucleolus of peer group games*, CentER DP, Tilburg University, (2003-19) (to appear in IJGT).

[5] R. Branzei, S. Tijs, J. Timmer, *Information collecting situations and bi-monotonic allocation schemes*, MMOR, 54, 303-313, (2001).

[6] R. van den Brink, *Null or zero players*, TI 2004-127/1, Tinbergen Institute DP, Amsterdam, (2004).

[7] R. van den Brink, G. van der Laan, V. Pruzhansky, *Harsanyi power solutions for graph-restricted games*, TI 2004-095/1, Tinbergen Institute DP, Amsterdam).

[8] B. Cailland, P. Jehiel, *Collusion in auctions with externalities*, The RAND J. of Econ., 29, 680-702, (1998).

[9] D. Graham, R. Marshall, *Collusive behavior at single-object second-price and English auctions*, J. of Political Econ. 95, 1217-31, (1987).

[10] D. Graham, R. Marshall, J.-F. Richard, *Differential payments within a bidder coalition and the Shapley value*, AER, 80, 493-510, (1990).

[11] K. Hendricks, R. H. Porter, G. Tan, *Joint bidding in federal off shore oil and gas lease auctions*, WP, Univ. B. C., (1999).

[12] K. Hendricks, R. H. Porter, *Collusion in auctions*, Annals d'Economic et de Statistique, 15/16, 217-230, (1989).

[13] P. Klemperer, *Auctions: Theory and Practice*, Princeton Univ. Press, (2004).

[14] V. Krishna, *Auction Theory*, Academic Press, (2002).

[15] G. J. Mailath, P. Zemsky, *Collusion in second price auctions with heterogeneous bidders*, GEB, 3, 467-486, (1991).

[16] R. C. Marshall, M. J. Meurer, *The economics of bidder collusion*, In: Chatterjee, Kalyan (ed.), Game theory and Business Applications, Norwell M. Kluwer, (2002).

[17] R. P. McAfee, J. McMillan *Bidding rings*, AER, 82, 579-599, (1992).

[18] P. Milgrom, *Putting Auction Theory to Work*, Cambridge University Press, (2004).

[19] S. Muto, *Sequential auctions on Böhm-Bawerks's horse market*, MASS 4, 117-130, (1983).

[20] S. Muto, M. Nakayama, J. Potters, S. Tijs, *On big boss games*, The economic studies quarterly, 39, 303-321 (1988).

[21] J. Potters, R. Poos, S. Tijs, S. Muto, *Clan Games*, GEB 1, 275-293.

[22] V. Reinhert, G. Belzer, *Some evidence on bid sharing and the use of information in the U.S. Treasury's auction experiment*, WP, Board of

Governers of the Federal Revenue System, (1996).

[23] M. S. Robinson, *Collusion and the choice of auction*, The RAND J. of Economics, 16, 141-145, (1985).

[24] A. Schotter, *Auctioning Böhm-Bawerk's horses*, IJGT, 3, 195-215, (1974).

[25] L. Shapley, *Cores of convex games*, IJGT, 1, 11-26, (1971).

[26] Y. Sprumont, *On parametric semi-infinite optimization*, Population monotonic allocation schemes for cooperative games with transferable utility, GEB2, 378-394, (1990).

[27] G. L. Thompson, *Auctions and market games*, (1981).

[28] S. Tijs, *Introduction to game theory*, Hindustan Book Agency, (2003).

[29] S. Tijs, A. Meca, M. Lòpez, *Benefit sharing in holding situations*, EJOR, 162, 251-269, (2005).

[30] W. Vickrey, *Counterspeculation, auctions and competitive sealed tenders*, Journal of Finance, 16, 8-37, (1961).

[31] W. Vickrey, *Auctions and bidding games*, In: Recent Advances in Game Theory, Princeton, N.J.: The Princeton University Conference, 15-27, (1962).

[32] R. Wilson, *Auction of shares*, Quarterly J. of Economics, 93, 675-689, (1979).